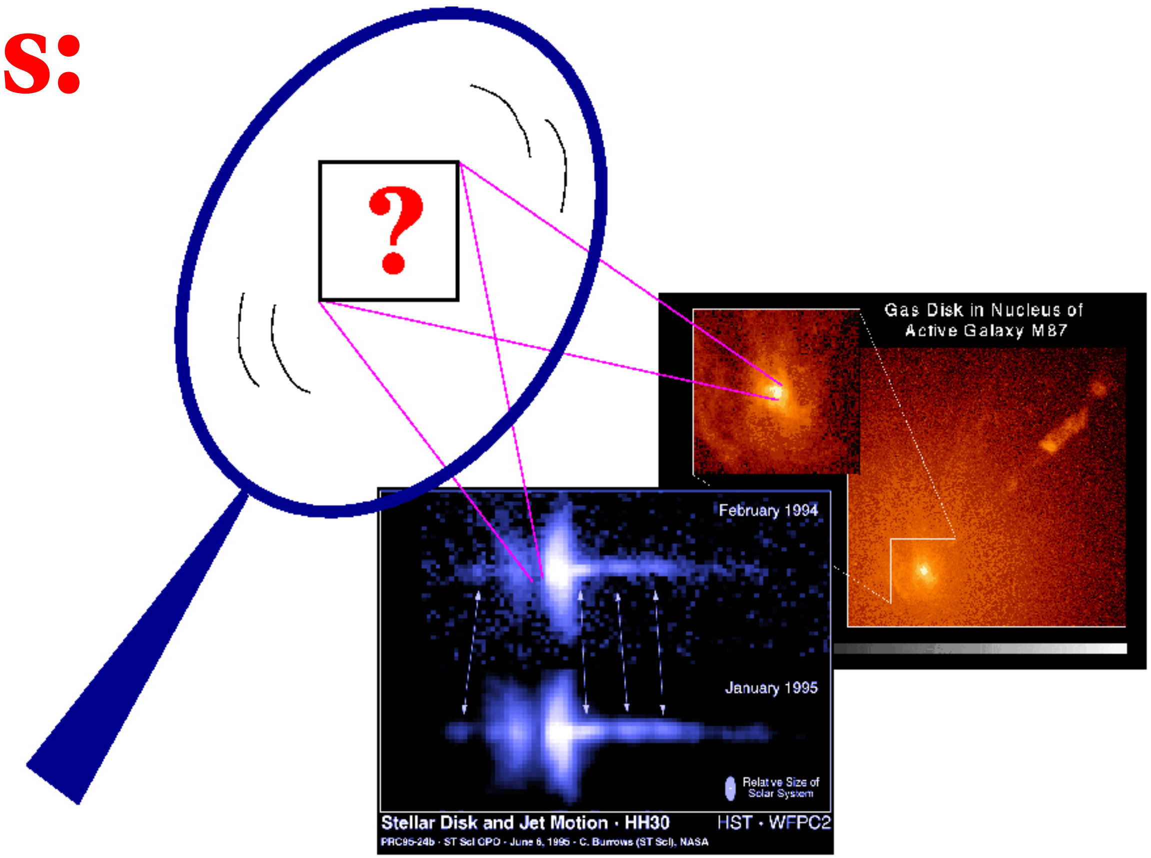
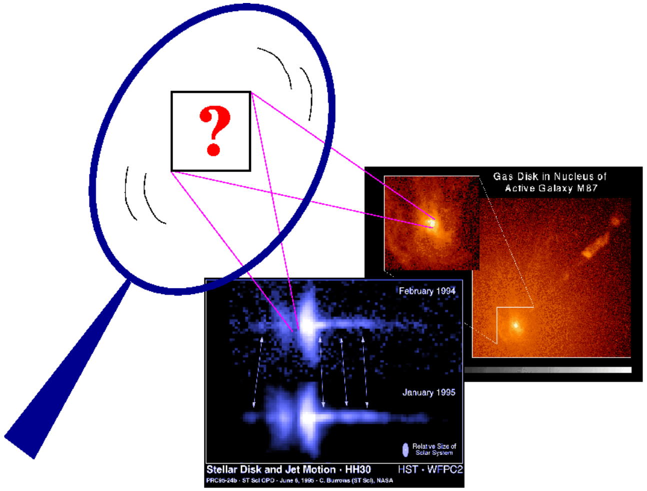
# Outflows & Jets: Theory & Observations




**Lecture winter term 2008/2009**

**Henrik Beuther & Christian Fendt**

# Outflows & Jets: Theory & Observations

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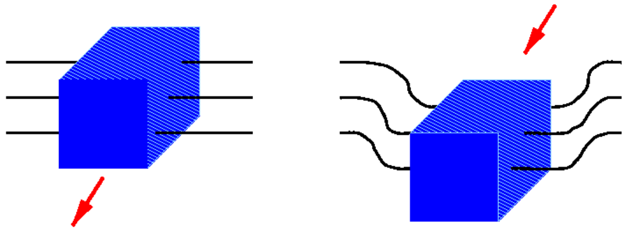
# Brief introduction to MHD

MHD concept: ionized, neutral, single fluid: average quantities: $\vec{j} \equiv q_{\rm e} \vec{v}_{\rm e} \rho_{\rm e} + q_{\rm i} \vec{v}_{\rm i} \rho_{\rm i}$

Ideal MHD: "frozen-in" field lines:

-> mass flux couples to magnetic flux

-> matter moves "along" the field lines



**MHD Lorentz force:** $\vec{F}_L \sim \vec{j} \times \vec{B}$

MHD equations (can only be solved numerically):

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0$$

$$\rho \left( \partial_t \vec{v} + (\vec{v} \cdot \nabla)\, \vec{v} \right) + \nabla P + \rho \nabla \Phi - \vec{j} \times \vec{B} = 0$$

$$\rho \left( \partial_t e + (\vec{v} \cdot \nabla)\, e \right) + P (\nabla \cdot \vec{v}) - \eta_{\rm D} |\vec{j}|^2 / c^2 = 0$$

$$\partial_t \vec{B} = \nabla \times \left( \vec{v} \times \vec{B} - \eta_{\rm D} \vec{j} / c \right)$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = 4\pi \vec{j} / c$$

(note non-ideal MHD resistive term of magnetic diffusivity)

# Brief introduction to MHD

MHD concept: ionized, neutral, single fluid: average quantities:

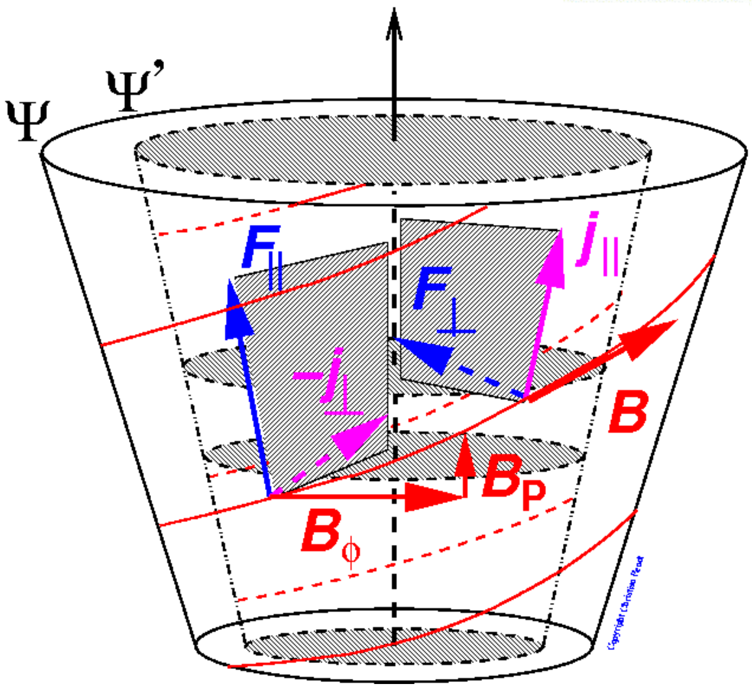
Ideal MHD: "frozen-in" field lines

**MHD Lorentz force:** $\vec{F}_L \sim \vec{j} \times \vec{B}$

**Axisymmetric jets:**

-> poloidal, toroidal field components: $B = B_p + B_\phi$

-> magnetic flux surfaces: $\Psi(R,Z) \sim \int \vec{B}_P \cdot d\vec{A}$




**Lorentz force components 1:**

projected on Ψ :

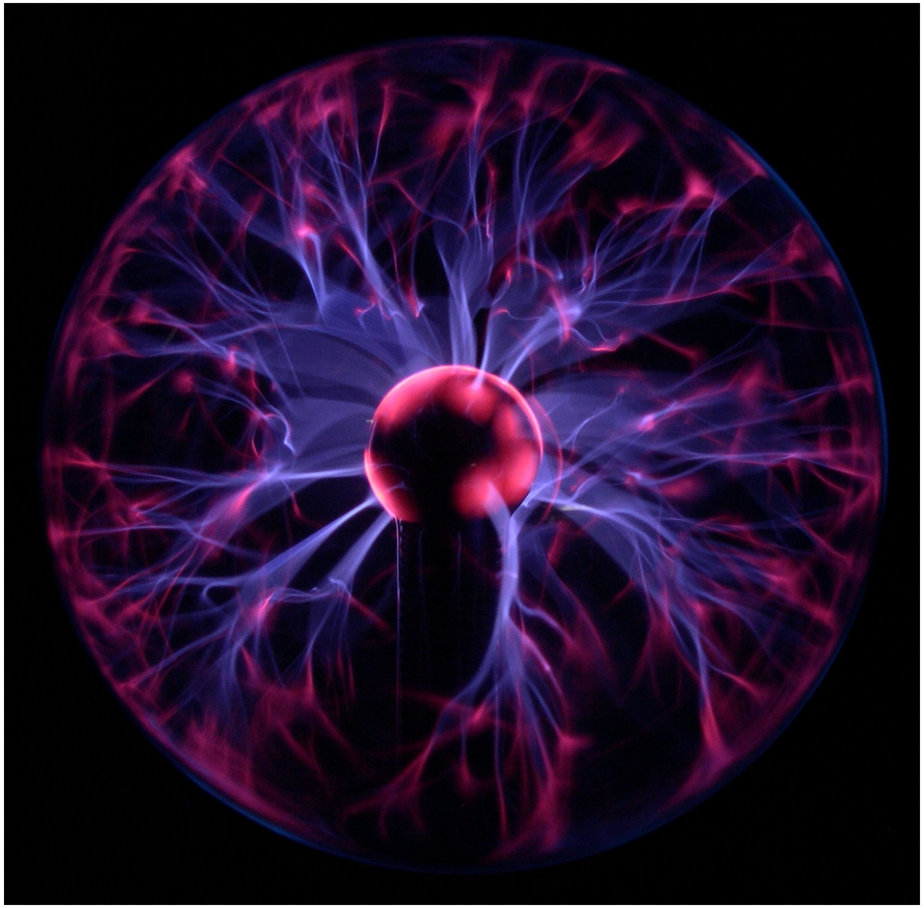
$$\vec{F}_L \equiv \vec{F}_{L,||} + \vec{F}_{L,\perp}$$

-> (de/) accelerating: $\vec{F}_{L,||} \equiv \vec{j}_\perp \times \vec{B}_\phi$

-> (de-) collimating: $\vec{F}_{L,\perp} \equiv \vec{j}_{||} \times \vec{B}$

-

# MHD theory



**Three approaches to describe an ionized gas**

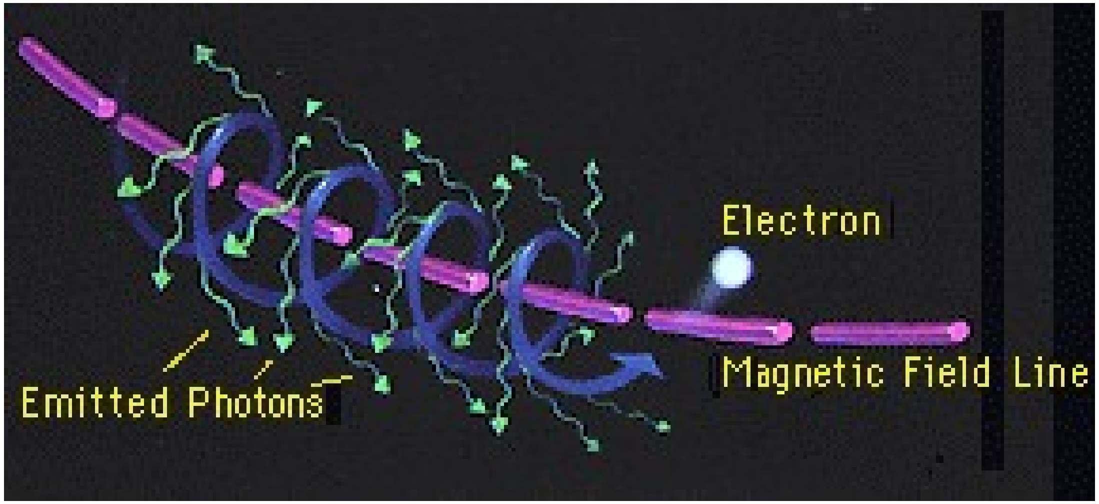
-> 1) test particles

-> 2) plasma physics (two fluid components)

-> 3) MHD (one fluid approach)

see e.g. : http://www.plasma-universe.com/index.php/Plasma-Universe.com

# MHD theory




## 1) ions & electrons

spiral along magnetic field

Lorentz force : $\vec{F} = q\,\vec{v} \times \vec{B}$

gyro radius / Larmor radius :

$$r_L \equiv v_{\tan} / \Omega_L$$

gyrofrequency/ cyclotron frequency : $\Omega_L = e\,B / m_{e,i}$

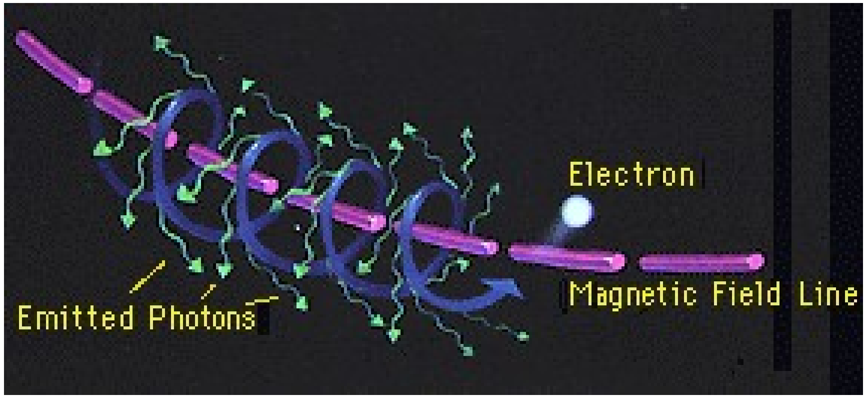
**->** gas is "magnetized" if characteristic length scale $L \gg r_L$

**->** **gas** moves on "straight" trajectories if $L \ll r_L$

-> magnetization parameter (e, i) : $\delta_{i,e} = r_L / L$

magnetized plasma if $\delta_i = r_{L,i} / L$

# MHD theory




## 1) ions & electrons

-> examples:

AGN jet (IC 4296): $B \sim 10\,\mu G\,, v \sim c_s \sim 0.005c\,, \rightarrow r_L = 0.03\,cm$

ISM, protons: $B \sim 3\,\mu G\,, 10^{17}\,eV\,, \rightarrow r_L = 30\ pc$

Galactic field, protons: $B \sim \mu G\,, 10^{19}\,eV\,, \rightarrow r_L = 3\,kpc$

-> UHE Cosmic Rays ~ $10^{21}$ eV, origin unknown;

particles escape if $L \ll r_L$

-> estimate of maximum energies in generation process

-> Radiation from gyrating articles: cyclotron & synchrotron emission, Bremsstrahlung

hot AGN jet plasma -> relativistic motion of particles **$\gamma$ < 1000**

-> compare to relativistic bulk motion of jet **$\Gamma$ < 10**

See Larmor radius calculator @ http://pps.coe.kumamoto-u.ac.jp/streaming/PulsedPower/formulary/cal-lr.html

# MHD theory

2) **plasma physics:** -> many particles -> statistical theory -> collective forces

see http://farside.ph.utexas.edu/teaching/plasma/lectures/lectures.html

**-> Quasi-neutrality:** number densities $n_i \sim n_e \sim n$

-> Plasma kinetic temperature (in energy units): $k\,T_{e,i} \equiv \frac{1}{3} m_{e,i} \langle v^2 \rangle$

thermal speed for $T_i = T_e = T$: $v_{th,e,i} \equiv \sqrt{2kT/m_{e,i}}$

**-> Plasma frequency:** collective dynamic behaviour: charge separation: $\sigma = e\,n\,\delta x = -\epsilon_0 E_x$

-> electrostatic oscillation in electric field: $m \frac{d^2}{dt^2} \delta x = e\,E_x = -m(\omega_P)^2 \delta x$

-> electron plasma frequency:

$$(\omega_{p,e})^2 = 4\pi n_e e^2 / m_e \quad , \quad \omega_{p,e}[s^{-1}] = 5.64 \times 10^4 n_e^{1/2} [cm^{-3}]$$

-> most fundamental time-scale in plasma physics

-> observable only if 1) oscillation period << life time of system:

2) external forcing slower than ω_p

-> if $L < v_{th} / \omega_p$ -> plasma behaviour not detected, particles escape

-> critical distance: **Debye length** $\lambda_D \equiv (kT/m)^{1/2} \omega_p^{-1} \ll L$ for a plasma

# MHD theory

## 2) plasma physics

-> **Debye shielding:** calculate average Coulomb force by charged particles:

-> Coulomb potential of test charge Q

-> without plasma: $\Phi = Q/r$

-> within plasma: polarization: charge density $\sigma = q\,(n' - n)$

(undisturbed and disturbed density of charges n and n')

-> Poisson equation: $\Delta\Phi = -4\pi\sigma - 4\pi Q\,\delta(r)$

-> thermodynamic equilibrium:

Boltzmann distribution of charged particles: $n' = n \exp(-q\Phi/kT)$

-> Boltzmann potential $\Phi$ should be local potential (not averaged)

$\langle \exp(-q\Phi/kT) \rangle \neq \exp(-q\langle\Phi\rangle/kT)$

-> Taylor expansion (far from charge Q): $\langle \exp(-q\Phi/kT) \rangle = 1 - \langle q\Phi/kT \rangle$

and $\sigma = -n q^2 \Phi/kT$

# MHD theory

## 2) plasma physics

-> solution of Poisson equation $$\frac{d^2}{dr^2}(r\Phi)=\frac{1}{r_D^2}(r\Phi)$$

with B.C. $\Phi \rightarrow Q/r$ for $r \rightarrow 0$

$\Phi \rightarrow 0$ for $r \rightarrow$ infinity

gives $$\Phi=(Q/r)\exp(-r/\lambda_D)$$

Debye length: $$\lambda_D=(kT/4\pi n q^2)^{1/2}$$

$$\lambda_D=743\,T^{1/2} n^{-1/2} cm$$

$$\lambda_D/r_{L,e}=220\,B\,n^{-1/2}$$

-> self-shielding distance, plasma charges "screen out" test charge

-> collective behaviour of particles, works if $\lambda_D \ll L$

# MHD theory

**2) plasma physics**

-> Debye number, Debye sphere: $N_D = n \frac{4\pi}{3} \lambda_D^3$

-> for N_D >> 1: collective behaviour; for N_D < 1: independent particles

-> plasma parameter : $\Lambda \equiv 1/N_D$

-> ionisation degree = relative number of ions and atoms: $\zeta = n_i / n_a$

-> depends on ionizing processes

-> in thermodynamic equilibrium: only temperature-dependent:: $\frac{n_i}{n_a} \sim g_e \exp(-\frac{\Phi_i}{kT})$

Saha equation: $$\frac{\zeta^2}{1-\zeta} \sim \frac{(kT\, m_e)^{3/2}}{n\, h^3} \exp(\Phi_I / kT)$$

-> $\zeta$ ~ 0.01% sufficient to behave as plasma

-> mean free path: $\lambda = v_{thermal} / v_{coll}$ , typical distance between collisions

collision-dominated plasma for $\lambda << L$, typically $\nu_{coll} \sim \omega_P (\ln \Lambda / \Lambda)$

collisionless plasma for $\lambda > L$, e.g. coupling by magnetic field

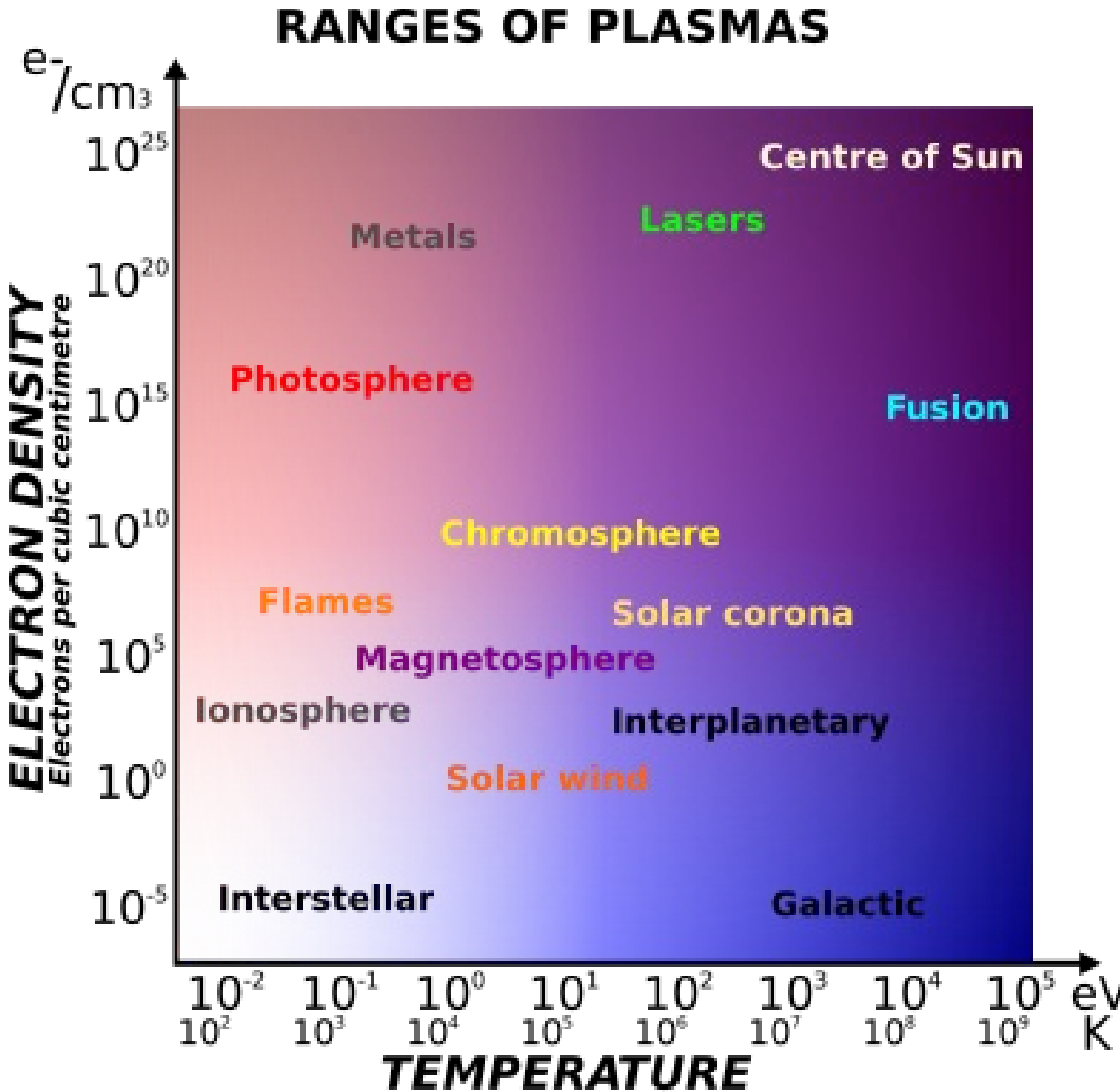
collisions -> help establishing Boltzmann distribution

$$\lambda[cm] = 1.44 \times 10^{13} (\ln \Lambda)\, n / (kT_e)^2 [eV, cm^{-3}]$$

# MHD theory

2) **plasma physics**

-> summary of parameters of different plasmas

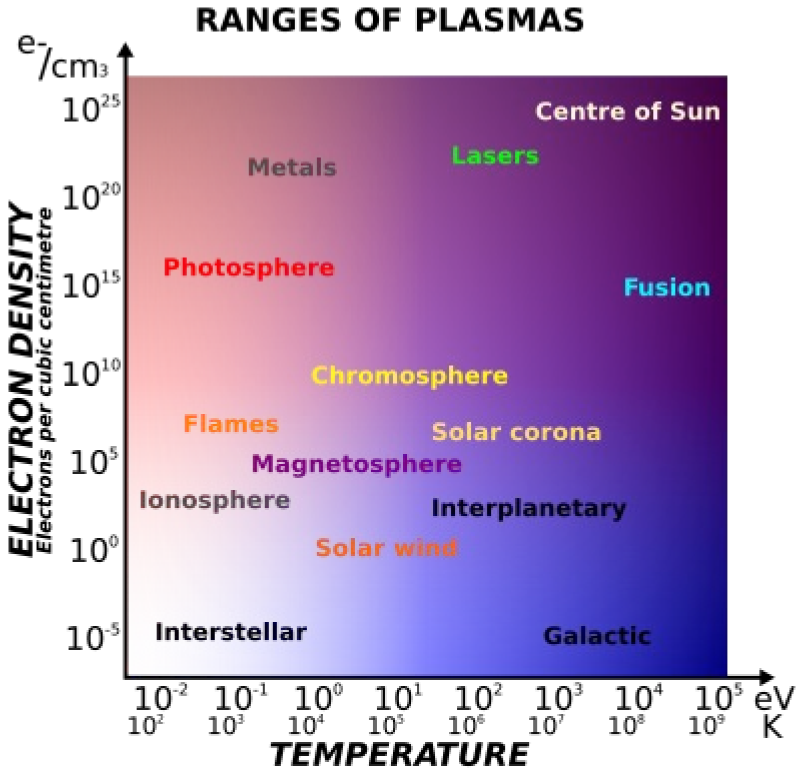



From ??, see Wiki

# MHD theory

2) **plasma physics**

-> summary of parameters of different plasmas




| Plasma | $n_e$ ($\mathrm{m}^{-3}$) | $T$ (K) | $B$ (T) | $\lambda_D$ (m) | $N_D$ | $\omega_p$ ($\mathrm{s}^{-1}$) | $\nu_{ee}$ ($\mathrm{s}^{-1}$) | $\omega_c$ ($\mathrm{s}^{-1}$) | $r_L$ ($m$) |
|---|---|---|---|---|---|---|---|---|---|
| Gas discharge | $10^{16}$ | $10^4$ | — | $10^{-4}$ | $10^4$ | $10^{10}$ | $10^5$ | — | — |
| Tokamak | $10^{20}$ | $10^8$ | 10 | $10^{-4}$ | $10^8$ | $10^{12}$ | $10^4$ | $10^{12}$ | $10^{-5}$ |
| Ionosphere | $10^{12}$ | $10^3$ | $10^{-5}$ | $10^{-3}$ | $10^5$ | $10^8$ | $10^3$ | $10^6$ | $10^{-1}$ |
| Magnetosphere | $10^7$ | $10^7$ | $10^{-8}$ | $10^2$ | $10^{10}$ | $10^5$ | $10^{-8}$ | $10^3$ | $10^4$ |
| Solar core | $10^{32}$ | $10^7$ | — | $10^{-11}$ | 1 | $10^{18}$ | $10^{16}$ | — | — |
| Solar wind | $10^6$ | $10^5$ | $10^{-9}$ | 10 | $10^{11}$ | $10^5$ | $10^{-6}$ | $10^2$ | $10^4$ |
| Interstellar medium | $10^5$ | $10^4$ | $10^{-10}$ | 10 | $10^{10}$ | $10^4$ | $10^{-5}$ | 10 | $10^4$ |
| Intergalactic medium | 1 | $10^6$ | — | $10^5$ | $10^{15}$ | $10^2$ | $10^{-13}$ | — | — |

# MHD theory



## 2) plasma physics: fluid model, kinetic theory

-> plasma physics: closure of Maxwell equations

by expressions for charge density $\rho_c$ and electric current density **j**

in terms of **E** and **B**

by microscopic distribution functions for each plasma species

-> define F_s(**r**, **v**, t) as microscopic phase-space density of plasma species s

near point (**r**,**v**) at time t.

F_s normalized to particle density in coordinate space: $\int \mathcal{F}_s(\mathbf{r}, \mathbf{v}, t)\, d^3\mathbf{v} = n_s(\mathbf{r}, t),$

-> phase space conservation requires: $\frac{\partial \mathcal{F}_s}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{F}_s + \mathbf{a}_s \cdot \nabla_v \mathcal{F}_s = 0,$

while $\mathbf{a}_s = \frac{e_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is acceleration of species s in **B** and **E**

-> averaging over ensemble: f_s = < F_s >_ensemble (**a** average plus collision operator):

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \bar{\mathbf{a}}_s \cdot \nabla_v f_s = C_s(f).$$

-> $C_s$ extremely complicated

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \bar{\mathbf{a}}_s \cdot \nabla_v f_s = 0.$$

-> for simplicity (Vlasov equation)

# MHD theory

## 2) plasma physics: kinetic theory, moments of distribution function

density (1st order): $n_s(\mathbf{r},t) = \int f_s(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{v},$

flux density (1st order ): $n_s \mathbf{V}_s(\mathbf{r},t) = \int \mathbf{v}\, f_s(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{v}.$ $V_s$ is flow velocity

charge density (1st order) : $\sum_s e_s n_s,$ electric current density (1st order): $\sum_s e_s n_s \mathbf{V}_s.$

stress tensor, momentum flow (2nd order): $\mathbf{P}_s(\mathbf{r},t) = \int m_s \mathbf{v}\mathbf{v}\, f_s(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{v}.$

energy flux density (3rd order): $\mathbf{Q}_s(\mathbf{r},t) = \int \frac{1}{2} m_s v^2 \mathbf{v}\, f_s(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{v}.$

heat flux density (rest frame): $\mathbf{q}_s(\mathbf{r},t) = \int \frac{1}{2} m_s w_s^{\,2} \mathbf{w}_s\, f_s(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{v}.$

pressure tensor (rest frame): $\mathbf{p}_s(\mathbf{r},t) = \int m_s \mathbf{w}_s \mathbf{w}_s\, f_s(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{v},$ $\mathbf{w}_s \equiv \mathbf{v} - \mathbf{V}_s,$

moments in diff. frames: $\mathbf{Q}_s = \mathbf{q}_s + \mathbf{p}_s \cdot \mathbf{V}_s + \frac{3}{2} p_s \mathbf{V}_s + \frac{1}{2} m_s n_s V_s^{\,2} \mathbf{V}_s.$

similar for collision operator ...

# MHD theory

**2)** plasma **physics: moments of kinetic equation, fluid equations**

**For each species ->** obtain fluid equations by taking moments of ensemble-avaraged

kinetic equation $\frac{\partial f_s}{\partial t} + \nabla \cdot (\mathbf{v} f_s) + \nabla_v \cdot (\mathbf{a}_s f_s) = C_s(f).$

continuity equation: $\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{V}_s) = 0.$

momentum conservation: $\frac{\partial (m_s n_s \mathbf{V}_s)}{\partial t} + \nabla \cdot \mathbf{P}_s - e_s n_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) = \mathbf{F}_s.$

energy cons.: $\frac{\partial}{\partial t}\left(\frac{3}{2} p_s + \frac{1}{2} m_s n_s V_s{}^2\right) + \nabla \cdot \mathbf{Q}_s - e_s n_s \mathbf{E} \cdot \mathbf{V}_s = W_s + \mathbf{V}_s \cdot \mathbf{F}_s.$

-> fluid equations by re-arrangement using pressure tensor,

heat flux density etc .... ----------------> e.g. $\frac{dn_s}{dt} + n_s \nabla \cdot \mathbf{V}_s = 0,$

-> closure of equations: express viscosity tensor, heat flux

density, collisional terms in terms of density, velocity, pressure ...

**-> hydrodynamic equations (for each species)**

# MHD theory

## 3) MHD equations

-> derived from two-fluid plasma equations under certain simplifications:

-> m_i >> m_e ; v_i ~ v_e ~ v_thermal ; charge neutrality

-> merge two-fluid equations to get one-fluid equation

example: velocities $\mathbf{V} = \dfrac{m_i \mathbf{V}_i + m_e \mathbf{V}_e}{m_i + m_e}, \quad \mathbf{j} = -ne\,\mathbf{U}, \quad \mathbf{U} = \mathbf{V}_e - \mathbf{V_i}$

-> $\mathbf{V}_i \simeq \mathbf{V} + O(m_e/m_i), \qquad \mathbf{V}_e \simeq \mathbf{V} - \dfrac{\mathbf{j}}{ne} + O\left(\dfrac{m_e}{m_i}\right).$

-> from that and $\dfrac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{V}_e)$ and $\dfrac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{V}_i)$ and $\nabla \cdot \mathbf{j} = 0$

follows continuity equation: $\dfrac{dn}{dt} + n \nabla \cdot \mathbf{V} = 0, \quad d/dt \equiv \partial/\partial t + \mathbf{V} \cdot \nabla$

-> similar for equation of motion (add two-fluid equations, total pressure p=pe+pi)

$$m_i n \frac{d\mathbf{V}}{dt} + \nabla p - \mathbf{j} \times \mathbf{B} \simeq 0.$$

# MHD theory

## 3) MHD equations

One-fluid equations + Maxwell equations; resistive; eq. of state needed for closure

**continuity equation**

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0$$

**equation of motion**

$$\rho \left( \partial_t \vec{v} + (\vec{v} \cdot \nabla) \, \vec{v} \right) + \nabla P + \rho \nabla \Phi - \vec{j} \times \vec{B} = 0$$

$$\rho \left( \partial_t e + (\vec{v} \cdot \nabla) \, e \right) + P (\nabla \cdot \vec{v}) - \eta_{\mathrm{D}} |\vec{j}|^2 / c^2 = 0$$

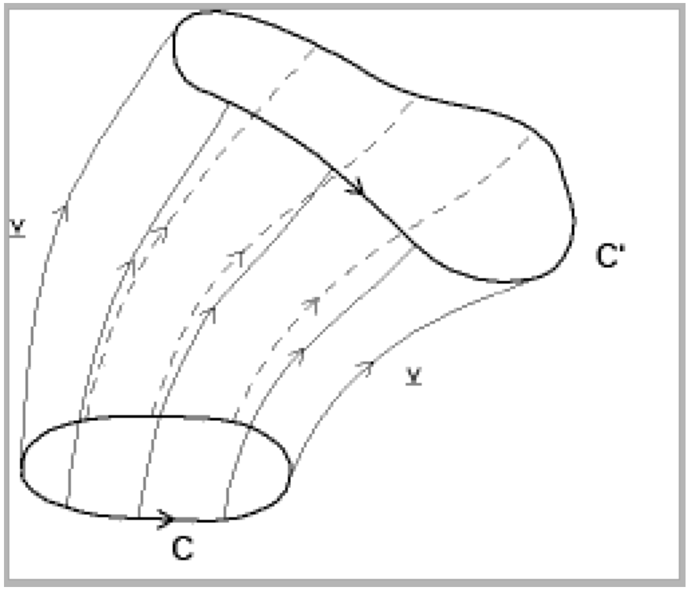
$$\partial_t \vec{B} = \nabla \times \left( \vec{v} \times \vec{B} - \eta_{\mathrm{D}} \vec{j} / c \right)$$

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{B} = 4\pi \vec{j} / c$$

**"no monopoles"** **Ampere's law** **induction equation**

**plus: equation od state,** eg. polytropic or isothermal gas

# MHD theory




## 3) MHD equations, flux freezing

Alfven's theorem (1943): *" In a perfectly conducting fluid, magnetic field lines move with the fluid: field lines are "frozen" into the plasma."*

-> A motion along magnetic field lines does not change the field, motions transverse to the field carry the field with them.

Integrate induction equation $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$. with Gauss' theorem $\int_V \nabla \cdot \mathbf{A}\, dV = \int_S \mathbf{A} \cdot d\mathbf{S},$

(**S** is closed surface enclosing volume **V**) and with Stokes' theorem $\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_C \mathbf{A} \cdot d\mathbf{l},$

(**C** is a closed curve around the open surface **S**; $d\mathbf{S} = \hat{\mathbf{n}}\, dS$ with the outward unit normal $\hat{\mathbf{n}}$)

(i) Since for all time $\nabla \cdot \mathbf{B} = 0$ -> $0 = \int_V \nabla \cdot \mathbf{B}\, dV = \int_S \mathbf{B} \cdot d\mathbf{S},$ $\forall t,$ (closed surface **S**)

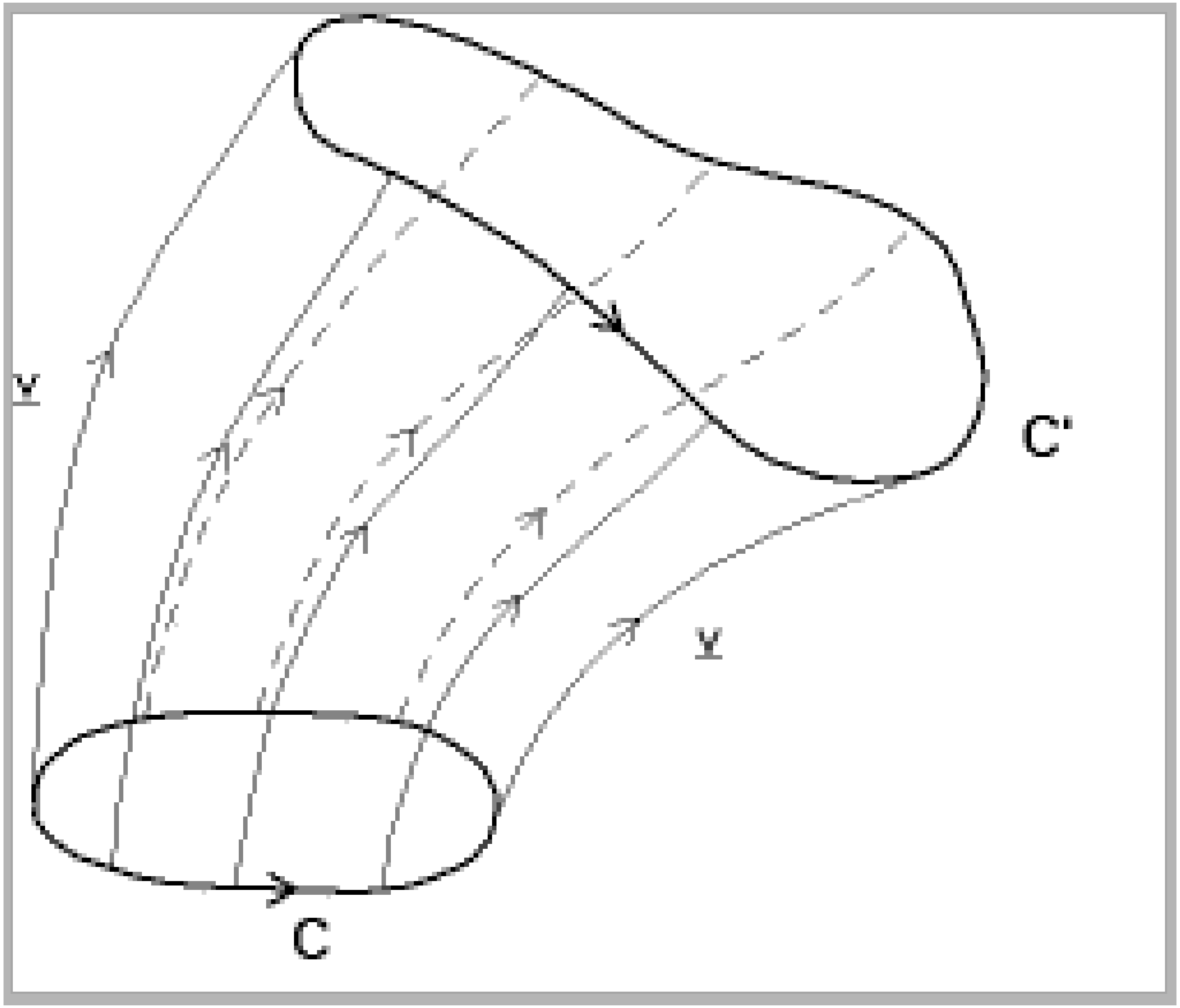
(ii) Time behaviour of the magnetic flux Φ through closed curve **C**, around an open surface **S1**:

$$\Phi = \int_{S_1} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{S}.$$

Φ changes in time since **B = B(t)** and since curve **C** changes in response to plasma motions.

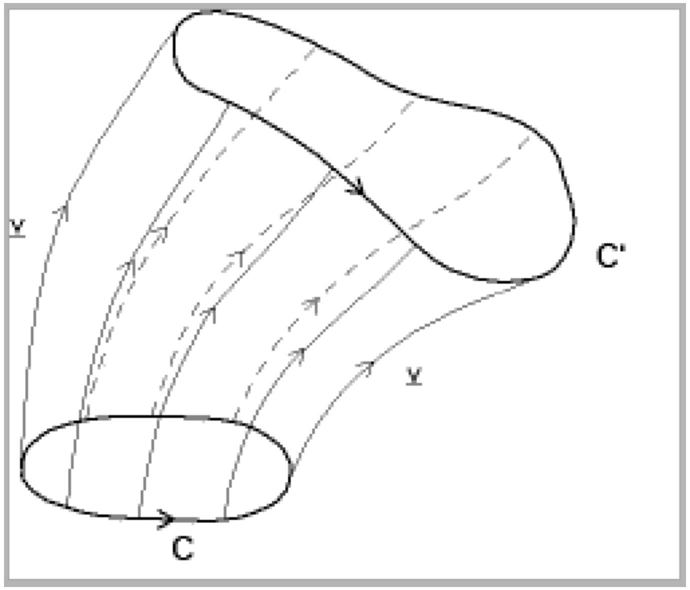
# MHD theory

## 3) MHD equations, flux freezing



# MHD theory

## 3) MHD equations, flux freezing




(iii) curve **C** moves with the fluid to curve **C '** within **δt**
motion of surface enclosed by **C** to surface enclosed by **C'** generates volume **V** enclosed by surface **S**

(iv) total flux through closed surface S in (iii): At time t+δt, when B(r, t+δt), we have

$$\begin{aligned} 0 &= \int_{closed S} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S}, \\ &= \int_{top} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S} + \int_{bottom} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S} + \int_{side} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S}, \\ &= \int_{C'} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S} - \int_{C} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S} + \int_{side} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S}. \end{aligned}$$

(v) Consider contribution to the total flux from curved side. A small element of length on the curve C traces out the shaded region. Then **dS** is given by the outward normal, $\hat{\mathbf{n}}$ times the area of shaded region. This area is approximately the area of parallelogram with sides **dl** and **vδt**. Hence, on the side **dS = dl x $\hat{\mathbf{n}}$ δt**. Thus,

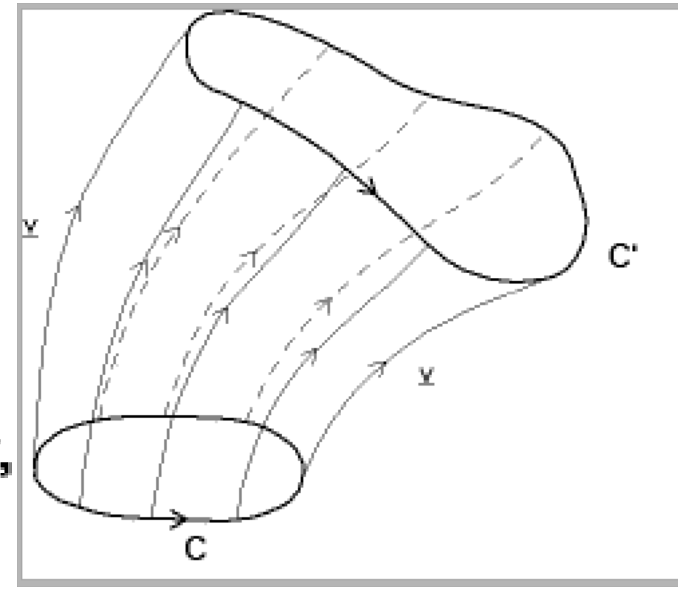
$$0 = \int_{C'} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S} - \int_{C} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S} + \int_{C} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{l}\times \mathbf{v}\delta t.$$

$$\int_{C'} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S} = \int_{C} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{S} - \delta t \int_{C} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot d\mathbf{l}\times \mathbf{v},$$

-> flux through curve **C'** at **t+δt** is equal to flux through curve **C** minus contribution from sides

# MHD theory

## 3) MHD equations, flux freezing




(vi) Change in flux in time is difference **Φ (t+δt) – Φ(t):**

$$\begin{aligned} \delta\Phi &= \Phi(t+\delta t) \text{ through } C' - \Phi(t) \text{ through } C, \\ &= \int_{C'} \mathbf{B}(\mathbf{r}, t+\delta t)\cdot \mathbf{dS} - \int_C \mathbf{B}(\mathbf{r}, t)\cdot \mathbf{dS}. \end{aligned}$$

With *** (2.41): $\delta\Phi = \int_C \left[\mathbf{B}(\mathbf{r}, t+\delta t) - \int_C \mathbf{B}(\mathbf{r}, t)\right]\cdot \mathbf{dS} - \delta t \int_C \mathbf{B}(\mathbf{r}, t+\delta t)\cdot \mathbf{dl}\times \mathbf{v}.$

Small **δt** -> approximate integrand in surface integral: $\mathbf{B}(\mathbf{r}, t+\delta t) - \int_C \mathbf{B}(\mathbf{r}, t)$-> $\delta t\, \partial \mathbf{B}/\partial t$

-> $\delta\Phi = \delta t \int_C \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{dS} - \delta t \int_C \mathbf{v}\times\mathbf{B}\cdot \mathbf{dl}.$ (identity $\mathbf{B}\cdot(d\mathbf{l}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{B}\times d\mathbf{l}) = (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}.$)

(vii) w/ induction eq.:
$$\begin{aligned} \frac{\delta\Phi}{\delta t} &= \int_C \nabla\times(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S} - \int_C \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}, \\ &= \int_C \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} - \int_C(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}, \text{ on using Stoke's theorem,} \\ &= 0. \end{aligned}$$

As **δt -> 0**, $\frac{\delta\Phi}{\delta t}\to\frac{d\Phi}{dt}$., thus **Φ** does not change in time, $\frac{d\Phi}{dt} = \frac{d}{dt}\left\{\int_C \mathbf{B}\cdot d\mathbf{S}\right\} = 0,$

where **C** is any closed contour moving with the fluid.

**=> Field lines are frozen into the plasma!**

# MHD theory

## MHD waves

-> define dynamical time scales // transport information / energy

-> linearize MHD equations, using **Q -> Q₀ + Q; Q₀:** equilibrium quantity, **Q:** perturbed q.

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{V} = 0 \qquad -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{V} \times \mathbf{B}_0) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{p}{p_0} - \frac{\Gamma \rho}{\rho_0}\right) = 0 \qquad \rho_0 \frac{\partial \mathbf{V}}{\partial t} + \nabla p - \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}_0}{\mu_0} = 0$$

-> look for wave-like solutions of linearized MHD equations, **Q ~ exp( k*r – ωt ) :**

$$-\omega\, \rho + \rho_0\, \mathbf{k} \cdot \mathbf{V} = 0 \qquad \omega\, \mathbf{B} + \mathbf{k} \times (\mathbf{V} \times \mathbf{B}_0) = 0$$

$$-\omega \left(\frac{p}{p_0} - \frac{\Gamma \rho}{\rho_0}\right) = 0 \qquad -\omega\, \rho_0\, \mathbf{V} + \mathbf{k}\, p - \frac{(\mathbf{k} \times \mathbf{B}) \times \mathbf{B}_0}{\mu_0} = 0$$

**--> $\rho(\rho_0,V)$, $p(p_0,V)$, $B(B_0,V)$**

-> substitute into linearized e.o.m.:

$$\left[\omega^2 - \frac{(\mathbf{k} \cdot \mathbf{B}_0)^2}{\mu_0\, \rho_0}\right] \mathbf{V} = \left\{\left[\frac{\Gamma p_0}{\rho_0} + \frac{B_0{}^2}{\mu_0\, \rho_0}\right] \mathbf{k} - \frac{(\mathbf{k} \cdot \mathbf{B}_0)}{\mu_0\, \rho_0} \mathbf{B}_0\right\} (\mathbf{k} \cdot \mathbf{V}) - \frac{(\mathbf{k} \cdot \mathbf{B}_0)\, (\mathbf{V} \cdot \mathbf{B}_0)}{\mu_0\, \rho_0} \mathbf{k}.$$

# MHD theory

## MHD waves

--> Define e.g. $\mathbf{B_o} \parallel \mathbf{e_z}$, wave-vector $\mathbf{k}$ in x-z plane, θ is angle between $\mathbf{B_o}$ and $\mathbf{k}$

--> eigenvalue equation:

$$\begin{pmatrix} \omega^2 - k^2 {V_A}^2 - k^2 {V_S}^2 \sin^2\theta & 0 & -k^2 {V_S}^2 \sin\theta\cos\theta \\ 0 & \omega^2 - k^2 {V_A}^2 \cos^2\theta & 0 \\ -k^2 {V_S}^2 \sin\theta \cos\theta & 0 & \omega^2 - k^2 {V_S}^2 \cos^2\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = 0.$$

Alfven speed $V_A = \sqrt{\frac{{B_0}^2}{\mu_0 \rho_0}}$ and sound speed $V_S = \sqrt{\frac{\Gamma p_0}{\rho_0}}$

--> eigenvalue equation solvable if determinant of square matrix vanishes

--> dispersion relation:

$$\left(\omega^2 - k^2 {V_A}^2 \cos^2\theta\right) \left[\omega^4 - \omega^2 k^2 \left({V_A}^2 + {V_S}^2\right) + k^4 {V_A}^2 {V_S}^2 \cos^2\theta\right] = 0.$$

# MHD theory

## MHD waves

Three independent roots:

1) $\omega = k\, V_A \cos\theta,$ eigenvector $(0, V_y, 0)$ , $\mathbf{k}\cdot\mathbf{V} = 0$ , $\mathbf{V}\cdot\mathbf{B}_0 = 0$

-> shear Alfven wave; no perturbation of plasma density; motion _|_ field

2) $\omega = k\, V_+,$ 3) $\omega = k\, V_-,$

with $V_\pm = \left\{ \frac{1}{2} \left[ {V_A}^2 + {V_S}^2 \pm \sqrt{({V_A}^2 + {V_S}^2)^2 - 4\, {V_A}^2 {V_S}^2 \cos^2\theta} \right] \right\}^{1/2}.$

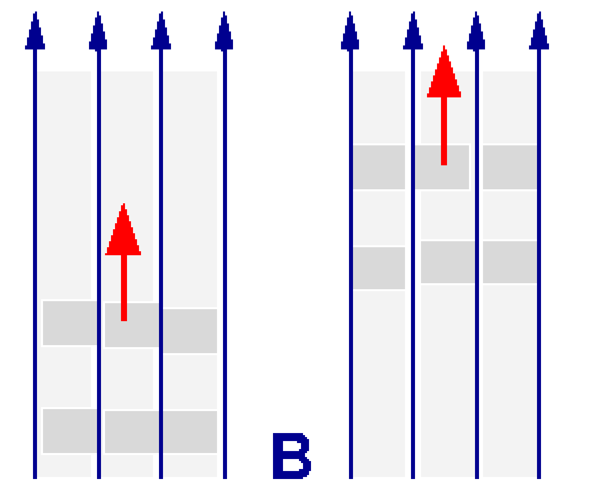
Note that $V_+ \geq V_-,$ eigenvector $(V_x, 0, V_z)$ $\mathbf{k}\cdot\mathbf{V} \neq 0,$ $\mathbf{V}\cdot\mathbf{B}_0 \neq 0$

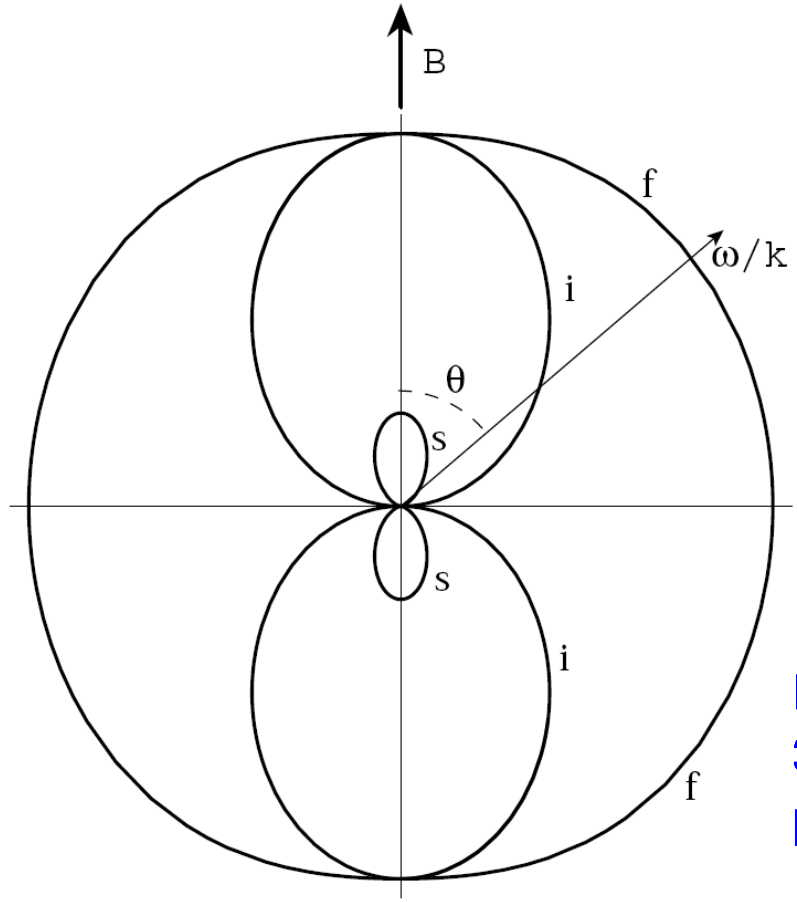
-> perturbations in density / pressure, motion || and _|_ to magnetic field
-> fast magnetosonic wave (2) and slow magnetosonic wave (3)

For limit $V_A \gg V_S$ (strong field), slow wave dispersion reduces, $\omega \simeq k\, V_S \cos\theta.$
-> sound wave along magnetic field lines

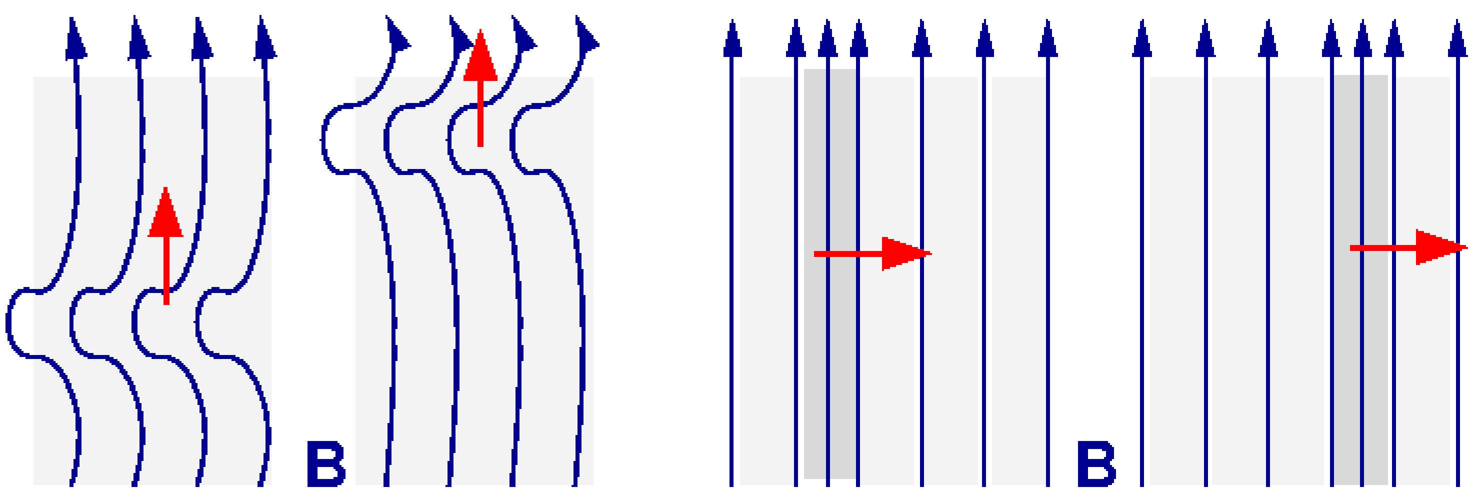
# MHD theory

## MHD waves




Phase velocities of 3 MHD waves; low plasma-β with $V_S < V_A$



# MHD theory

## Stationary axisymmetric MHD

First (?) presented by Chandrasekhar (1956):

-> Interesting setup for **jets and outflows** -> essential properties drop out naturally

-> Example: derivation for Ferraro's law of isorotation:

1) use cylindrical coordinates (r,ϕ,z)

2) decompose vectors in poloidal (meridional plane), & toroidal components (ϕ-component)

3) stationary Faraday's law: $\nabla \times \vec{E} = 0$ --> potential field: $\vec{E} = \nabla U$

4) axisymmetry: $E_\phi = 0$

5) infinite conductivity: Ohms law: $\vec{E} = -\frac{1}{c}\vec{v} \times \vec{B}$

6) since $E_\phi = 0$ --> $\vec{v}_p \times \vec{B}_p = 0$ or $\vec{v}_p || \vec{B}_p$ $\rightarrow \vec{v}_p = \kappa(R,Z)\vec{B}_p$

--> poloidal velocity parallel to poloidal field

7) mass conservation & stationarity --> $\nabla(\rho\vec{v}_p) = 0$

8) for $\kappa(R,Z)$ -> $0 = \nabla(\rho\kappa\vec{B}_p) = B_p \cdot \nabla(\rho\kappa)$ -> $\eta \equiv \rho\kappa$ concerved along field lines

9) introduce magnetic flux function: $\Psi(R,Z) = \frac{1}{2\pi}\int \vec{B}_p \cdot d\vec{A} = \frac{1}{2\pi}\int B_z\, R\, d\phi\, dR,$

# MHD theory

## Stationary axisymmetric MHD

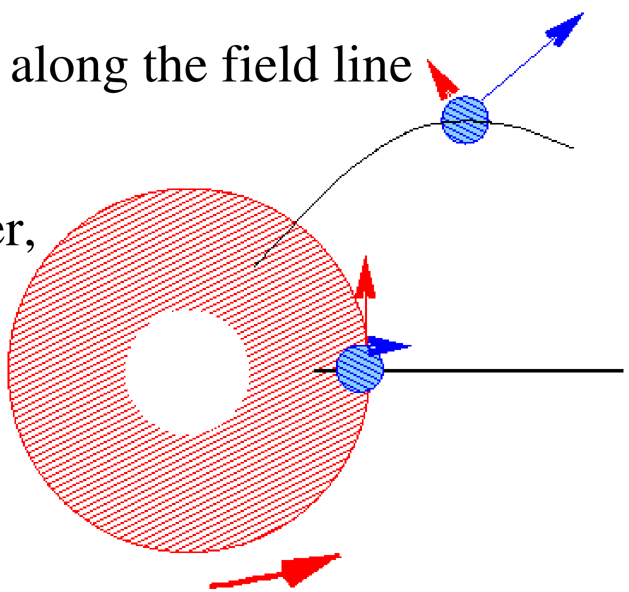
-> exemple: derivation for Ferraro's law of isorotation

9) Introduce magnetic flux function: $\Psi(R,Z) = \frac{1}{2\pi}\int \vec{B}_p \cdot d\vec{A} = \frac{1}{2\pi}\int B_z\, R\, d\phi\, dR,$

10) We have further $\vec{v} \times \vec{B} = \vec{v}_\phi \times \vec{B}_p + \vec{v}_p \times \vec{B}_\phi = \frac{1}{R}\left(v_\phi - \frac{\eta}{\rho}B_\phi\right)\nabla\Psi$

11) With MHD condition 5) and Faraday's law: $0 = \nabla \times \vec{E} = \nabla\Psi \times \nabla\left(\frac{1}{R}\left(v_\phi - \frac{\eta}{\rho}B_\phi\right)\right)$

12) Thus the quantity $\Omega_F \equiv \left(v_\phi - \frac{\eta}{\rho}B_\phi\right)/R$ is conserved along the field line



-> **Ferraro's law of isorotation,** iso-rotation parameter,
"angular velocity of field line"

# MHD theory

## Stationary axisymmetric MHD

Conservation laws of stationary MHD:

Mass flow rate per flux surface:

$$\eta(\Psi) \equiv \rho \frac{v_p}{B_p} \mathrm{sgn}(\vec{v}_p \cdot \vec{B}_p)$$

$$\dot{M}(\Psi + \Delta\Psi) - \dot{M}(\Psi) = \int_{\Psi}^{\Psi+\Delta\Psi} \rho \vec{v}_p \cdot d\vec{A} = \int_{\Psi}^{\Psi+\Delta\Psi} \eta \vec{B}_p \cdot d\vec{A}.$$

Field line iso-rotation:

$$\Omega_F(\Psi) \equiv \frac{1}{R} \left( v_\phi - \frac{\eta(\Psi)}{\rho} B_\phi \right)$$

Energy conservation:

$$E(\Psi) = \frac{v^2}{2} - \frac{R B_\phi \Omega_F(\Psi)}{4\pi\eta(\Psi)}$$

Angular momentum conservation:

$$L(\Psi) = R^2 \Omega_F(\Psi) - \frac{R B_\phi}{4\pi\eta(\Psi)}$$

# Stationary MHD - the solar wind

## Example: Parker wind

Parker's (1958) prediction: solar corona not in static equilibrium -> expansion
(later confirmed by satellite missions)

Parker model of solar wind: stationary, spherically symmetric, hydrodynamic, isothermal

-> mass conservation: $$\nabla \cdot (\rho \mathbf{v}) = \mathbf{0}$$

-> momentum conservation: $$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \mathbf{P} + \rho \mathbf{g}$$

-> assumptions: $$P = \rho \mathcal{R} T,\ T = T_o,\ \mathbf{v} = \mathbf{v}\dot{\mathbf{r}},\quad g = -GM_o/r^2$$

-> mass conservation: $$\frac{d}{dr}(r^2 \rho v) = 0 \longrightarrow\longrightarrow r^2 \rho v = const.$$

-> radial momentum conservation: $$\rho v \frac{dv}{dr} = -\frac{dP}{dr} - \frac{GM_O \rho}{r^2}$$

-> applying isothermal sound speed: $$c_s \equiv (P/\rho)^{1/2},\ P = c_s^2 \rho$$

---> $$\rho v \frac{dv}{dr} = -c_s^2 \frac{d\rho}{dr} - \frac{GM_O \rho}{r^2}$$

# Stationary MHD - the solar wind

## Example: Parker wind

-> applying mass conservation ->

$$v\frac{dv}{dr} = -c_s^2 r^2 v \frac{d}{dr}\left(\frac{1}{r^2 v}\right) - \frac{GM_o}{r^2}$$

-> wind equation ->

$$\left(1 - \frac{c_s^2}{v^2}\right) v\frac{dv}{dr} = 2\frac{c_s^2}{r^2}\left(r - r_s\right)$$

-> integrate analytically / numerically:

-> critical wind solution:

critical point:

$$r_s = GM_o/c_s^2$$

-> v = cs

-> critical point = sonic point

-> solar parameters:

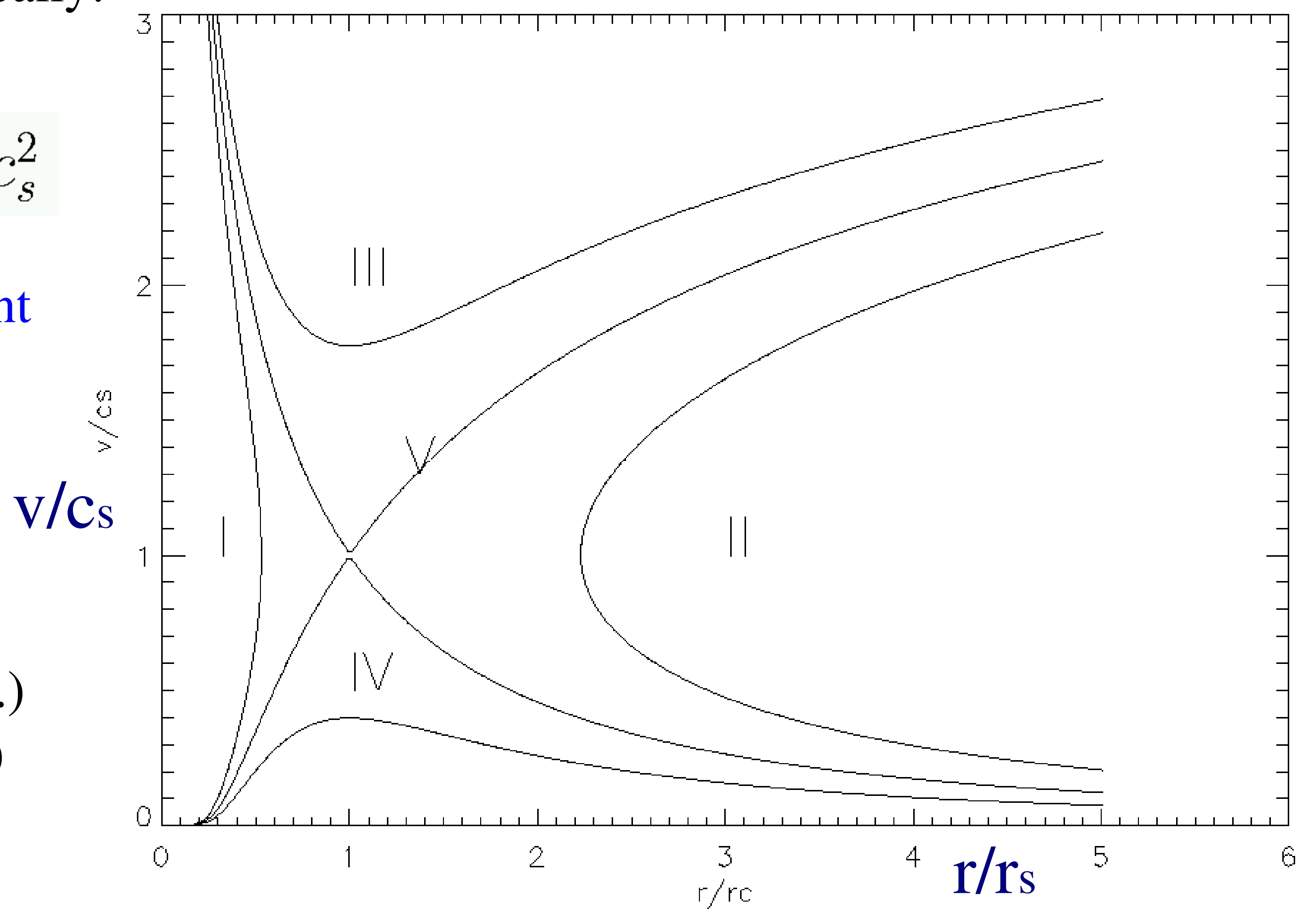
-> T = 10^6 (corona)

cs = 100 km/s

rs = 10 ro

v(1AU) = 310 km/s (pred.)

v(1AU) = 320 km/s (obs.)



# Stationary MHD - the solar wind

## Example: Solar wind / Weber & Davis (1967)

-> magnetized solar wind; magnetic wind equation: Alfven Mach number M_A = v / v_A
-> stellar rotation essential; magnetic field removes angular momentum -> stellar braking
-> radial momentum conservation:

$$\frac{d}{dr}\left\{\frac{1}{2}u^2+\frac{\gamma}{\gamma-1}\frac{p_a}{\rho_a}\left(\frac{\rho}{\rho_a}\right)^{\gamma-1}-\frac{GM_\odot}{r}\right\}=\frac{v_\phi^2}{r}-\frac{1}{8\pi\rho r^2}\frac{d}{dr}(rB_\phi)^2$$

-> magnetic wind equation:

$$\frac{du}{dr}=\frac{u}{r}\left\{\left(\frac{2\gamma p_a}{\rho_a M_A^{2(\gamma-1)}}-\frac{GM_\odot}{r}\right)(M_A^2-1)^3+\Omega^2 r^2\left(\frac{u}{u_a}-1\right)\left[(M_A^2+1)\frac{u}{u_a}-3M_A^2+1\right]\right\}$$
$$\times\left[\left(u^2-\frac{\gamma p_a}{\rho_a M_A^{2(\gamma-1)}}\right)(M_A^2-1)^3-\Omega^2 r^2 M_A^2\left(\frac{r_a^2}{r^2}-1\right)^2\right]^{-1}$$

-> additional critical points / singularities:

-> slow magnetosonic point / Alfven point / fast magnetosonic point
determine critical solution:
-> v = v_sm at r= r_sm; v = v_A at r = r_A; v = v_fm at r = r_fm

# Stationary MHD - relativistic wind

## Example: MHD wind in Kerr metric

--> relativistic treatment of motion and metric:

relativistically defined velocity: $u_{\rm p} \equiv \gamma v_{\rm p}/c,$

--> wind equation:

$$u_{\rm p}^2 + 1 = -\sigma_{\rm m} \left(\frac{E}{\mu}\right)^2 \frac{k_0 k_2 + \sigma_{\rm m} 2 k_2 M^2 - k_4 M^4}{(k_0 + \sigma_{\rm m} M^2)^2},$$

metric:

$$k_0 = g_{33}\Omega_{\rm F}^2 + 2 g_{03}\Omega_{\rm F} + g_{00},$$

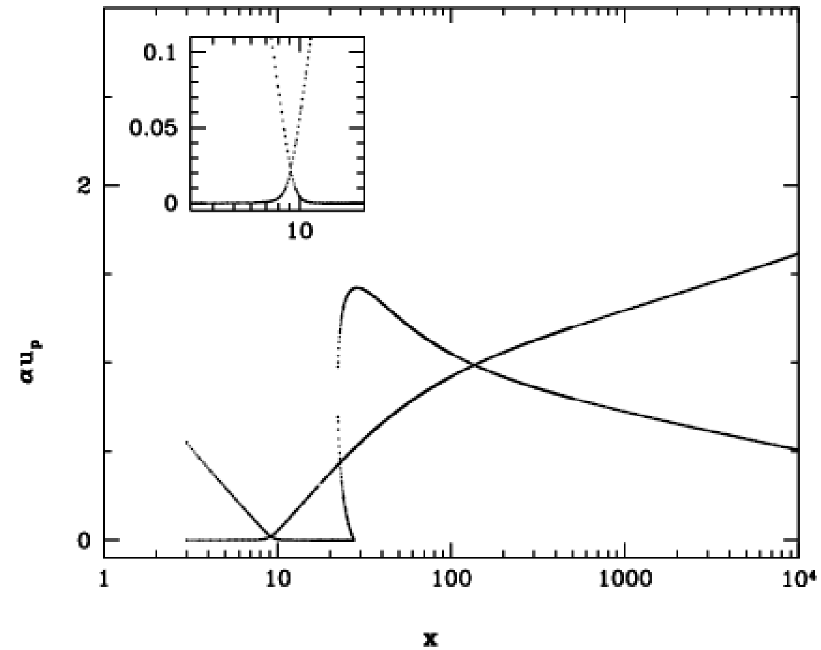
$$k_2 = 1 - \Omega_{\rm F}\frac{L}{E},$$

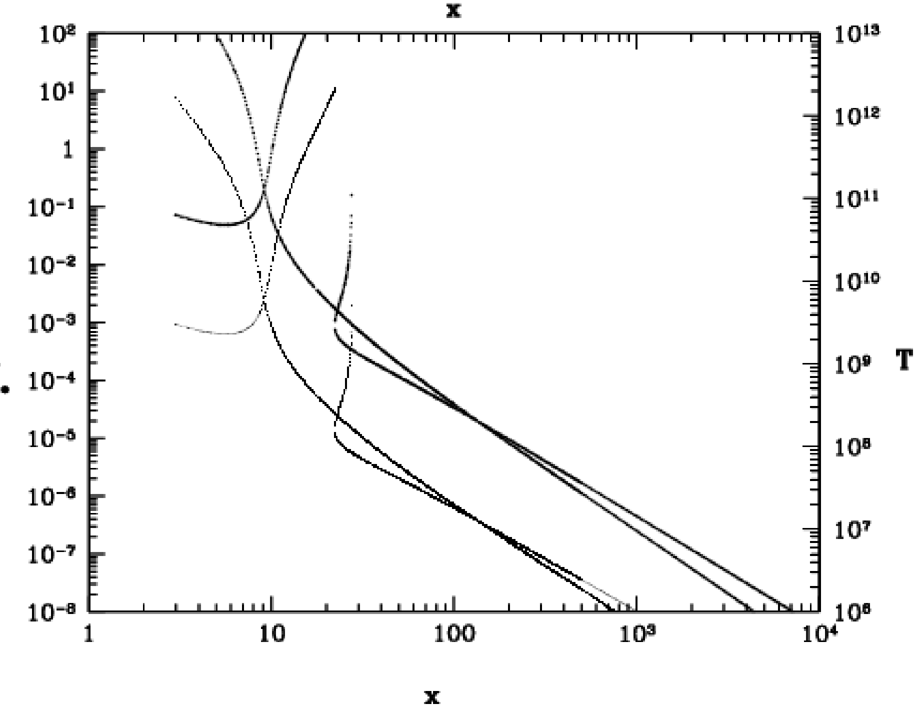
$$k_4 = -\left(g_{33} + 2 g_{03}\frac{L}{E} + g_{00}\frac{L^2}{E^2}\right) / \left(g_{03}^2 - g_{00} g_{33}\right)$$

-> 3 critial points

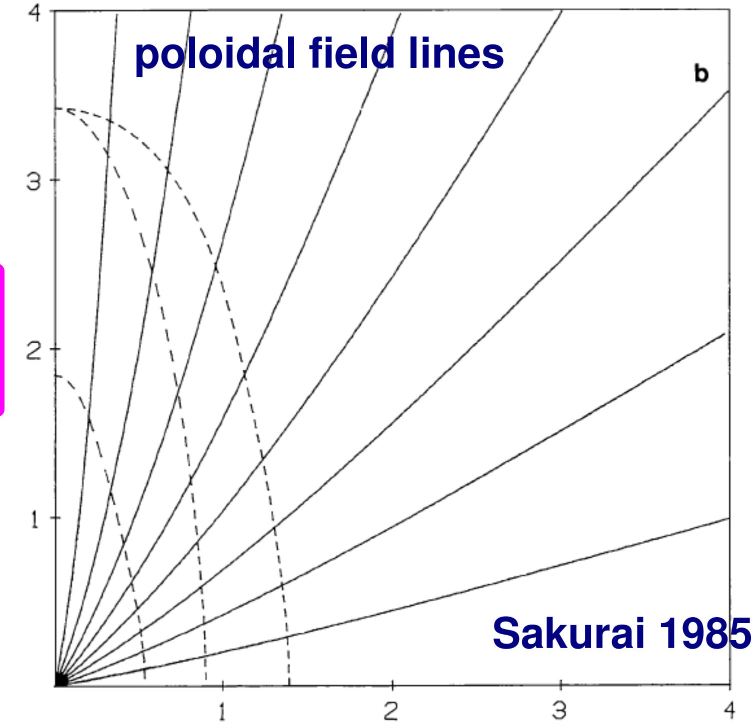
-> (highly) relativistic velocities

-> MHD condition applicable ??



# Stationary MHD - stellar winds




## Axisymmetric structure of MHD jets:

**-> dynamics along a given field line by wind equation**

**-> structure magnetic field:**

force-balance across the field / flux surfaces

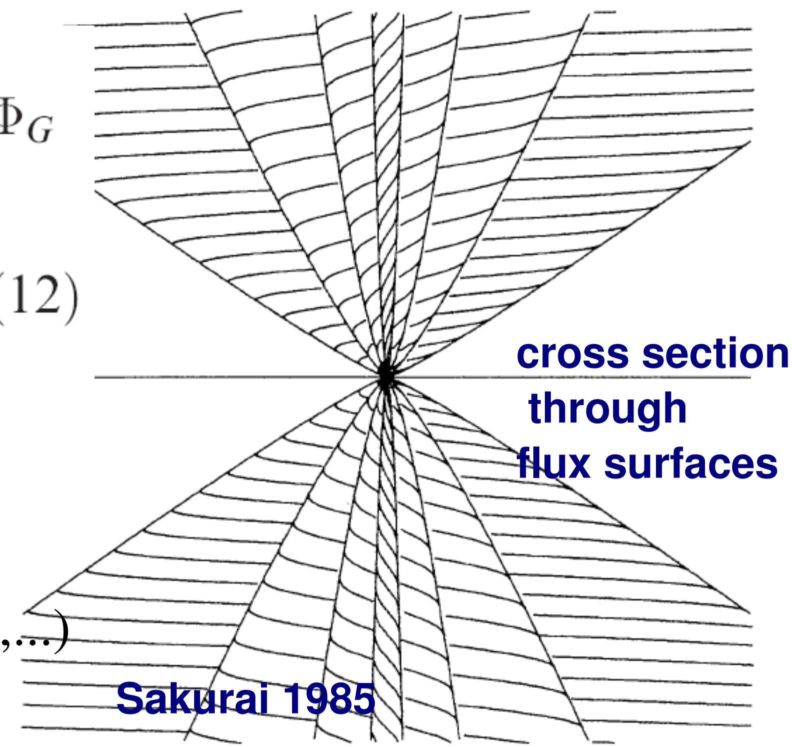
-> project eq. of motion to magnetic surfaces **a(r,z)** :

- consider **ϕ**-component of Ampere's law
- take current density from eq. of motion

-> Grad-Shafranov (GS) equation: (curvature Ψ)

$$\left(1-\frac{\rho}{\rho_{\rm A}}\right)v_P^2\frac{d\psi}{ds}=\frac{1}{\rho}\frac{\boldsymbol{\nabla}a}{|\nabla a|}\cdot\boldsymbol{\nabla}\left(\frac{|\boldsymbol{\nabla}a|^2}{2\mu_0 r^2}+Q\rho^\gamma\right)+\frac{\boldsymbol{\nabla}a}{|\nabla a|}\cdot\boldsymbol{\nabla}\Phi_G$$

$$-\left(L-\frac{I}{\alpha}\right)^2\frac{1}{r^3}\frac{\partial a/\partial r}{|\nabla a|}+\frac{1}{\rho r^2}\frac{\boldsymbol{\nabla}a}{|\nabla a|}\cdot\boldsymbol{\nabla}\left(\frac{\mu_0 I^2}{2}\right)\ . \qquad (12)$$




-> r.h.s.: poloidal magnetic pressure gradient, gas pressure gradient, gravity, centrifugal force, hoop stress

-> GS contains dynamical parameters (density, velocity,...)

-> to be calculated by wind equation

-> iterative procedure for solution (Sakurai 1985)

Outflows & Jets: Theory & Observations

# MHD theory -- simulations

$$\partial_t \rho + \nabla\cdot(\rho\vec{v}) = 0$$

$$\rho\left(\partial_t \vec{v} + (\vec{v}\cdot\nabla)\,\vec{v}\right) + \nabla P + \rho\nabla\Phi - \vec{j}\times\vec{B} = 0$$

$$\rho\left(\partial_t e + (\vec{v}\cdot\nabla)\,e\right) + P(\nabla\cdot\vec{v}) - \eta_{\rm D}|\vec{j}|^2/c^2 = 0$$

$$\partial_t \vec{B} = \nabla\times\left(\vec{v}\times\vec{B} - \eta_{\rm D}\vec{j}/c\right)$$

$$\nabla\cdot\vec{B} = 0, \quad \nabla\times\vec{B} = 4\pi\vec{j}/c$$

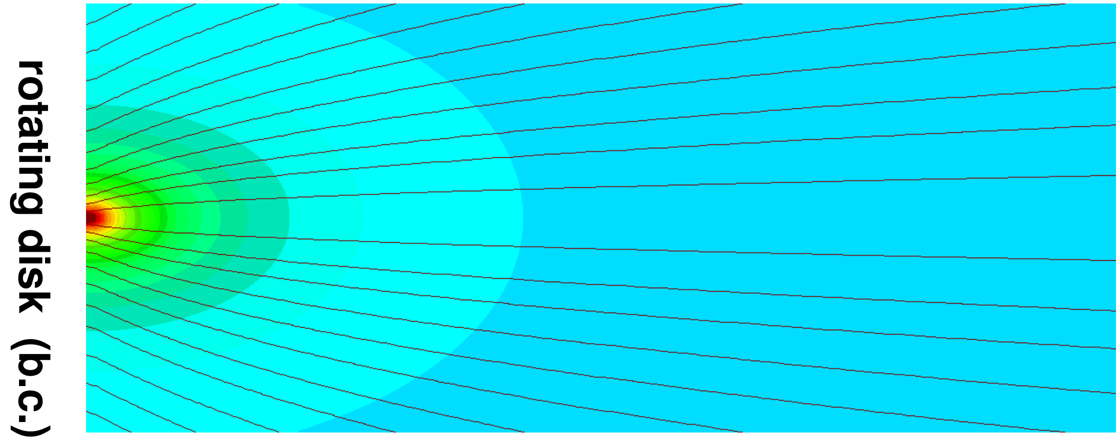
**Time-dependent solutions**
**of MHD equations**
**by numerical simulations**

-> Numerical MHD codes:
ZEUS, Flash, Pluto, Nirvana ...

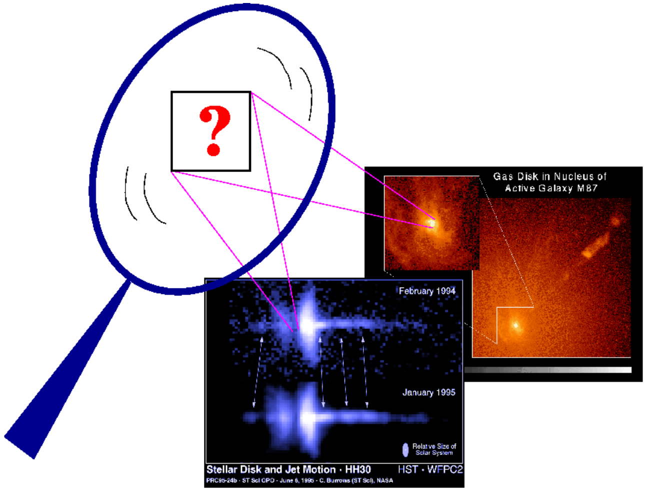
-> apply astrophysical boundary conditions (disk/stellar magnetic field, mass flow rates ....)

-> some advantages:
- time-dependent evolution
- no need to search for critical solutions
- 3D solutions possible
- inclusion of more physics "simple" (radiation losses, turbulent viscosity,resistivity ...)

--> some difficulties:
- dynamic range (strong density contrast, strong gradients)
- computer power limited (grid size, time resolution)



# Outflows & Jets: Theory & Observations

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17.10 Definitions, parameters, basic observations (H.B.)
24.10 Basic theoretical concepts & models I (C.F.): Astrophysical models, MHD
31.10 Basic theoretical concepts & models II (C.F.) : MHD, derivations, applications
*07.11 Observational properties of accretion disks (H.B.)*
14.11 Accretion disk theory and jet launching (C.F.)
21.11 Outflow-disk connection, outflow entrainment (H.B.)
28.11 Outflow-ISM interaction, outflow chemistry (H.B.)
05.12 Theory of outflow interactions; Instabilities (C.F.)
12.12 Outflows from massive star-forming regions (H.B.)
19.12 Radiation processes - 1 (C.F.)
*26.12 and 02.01 Christmas and New Year's break*
09.01 Radiation processes - 2 (H.B.)
16.01 Observations of AGN jets (C.F.)
23.01 Some aspects of AGN jet theory (C.F.)
30.01 Summary, Outlook, Questions (H.B. & C.F.)