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ASTRONOMY AND ASTROPHYSICS

Differentially rotating relativistic magnetic jets

Asymptotic trans-field force-balance including differential rotation

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Abstract. Highly collimated jets are observed in various astronomical objects, as active galactic nuclei, galactic high energy sources, and also young stellar objects. There is observational indication that these jets originate in accretion disks, and that magnetic fields play an important role for the jet collimation and plasma acceleration. The rapid disk rotation close to the central object leads to relativistic rotational velocities of the magnetic field lines.

The structure of these axisymmetric magnetic flux surfaces follows from the trans-field force-balance described by the Grad-Schlüter-Shafranov equation. In this paper, we investigate the asymptotic field structure of differentially rotating magnetic jets, widening the study by Appl & Camenzind (1993a,b).

In general, our results show that, with the same current distribution, differentially rotating jets are collimated to smaller jet radii as compared with jets with rigidly rotating field. Differentially rotating jets need a stronger net poloidal current in order to collimate to the same asymptotic radius. Current-free solutions are not possible for differentially rotating *disk*-jet magnetospheres with cylindrical asymptotics.

We present a simple analytical relation between the poloidal current distribution and magnetic field rotation law. A general relation is derived for the current strength for jets with maximum differential rotation and minimum differential rotation. Analytical solutions are also given in the case of a field rotation leading to a degeneration of the light cylinder.

By linking the asymptotic solution to a Keplerian accretion disk, 'total expansion rates' for the jets, and also the flux distribution at the foot points of the flux surfaces are derived. Large poloidal currents imply a strong opening of flux surfaces, a stronger gradient of field rotation leads to smaller expansion rates. There is indication that AGN jet expansion rates are less than in the case of protostellar jets. High mass AGN seem to have larger jet expansion rates than low mass AGN.

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1. Jet formation from disk magnetic fields

Observations of different kinds of jet sources give convincing evidence that jet formation is always connected to the presence of an accretion disk. This holds for various scales of energy output, jet velocity and nature of the jet emitting objects as there are active galactic nuclei (AGN), galactic superluminal jet sources, mildly relativistic jets from neutron stars (e.g. SS 433), and the numerous class of protostellar jets (e.g. Zensus et al. 1995; Mirabel & Rodriguez 1995; Mundt et al. 1990, Ray et al. 1996).

It is now generally accepted that magnetic fields play an important role in jet formation and propagation for all different kinds of jet sources. These jets are believed to originate very close to the central objects in the interaction region with the accretion disk or in the disk itself.

If the central object is a black hole as it is likely for AGN and galactic superluminal jet sources, the disk is the only possible location for a field generation (by dynamo action or/and advection of magnetic flux).

In the case of protostars and neutron stars the central object also carries a relatively strong magnetic field, and it is not yet clear, whether the jet magnetic field originates in the disk or in the star. However, there must clearly be a strong interaction between the stellar field and the accretion flow in a region, where the stellar field couples to the disk.

Plasma is ejected from the disk into the magnetosphere and becomes magnetically accelerated (see Ferreira & Pelletier 1995). Electric currents and inertia associated with the plasma flow collimate the jet. The observed degree of collimation is very high. The extragalactic jets, the galactic superluminal jets as well as protostellar jets are collimated almost to a cylindrical shape (Camenzind & Krockenberger 1992, Zensus et al. 1995; Ray et al. 1996).

While for extragalactic and galactic superluminal jets a fully relativistic description is obviously necessary, the case of protostellar jets is more complicated. The protostellar jet velocities of about $\simeq 400\,\mathrm{km\,s^{-1}}$ (Mundt et al. 1990) are clearly non-relativistic. However, if the field is anchored in the accretion disk, the rapid rotation of the inner disk may lead to field rotational velocities of the order of the speed of light (Camenzind 1990; see also Fendt et al 1995). In this case a relativistic treatment of the MHD would be required. We emphasise that there are no relativistic effects in the dynamics of the jet motion itself (since the Alfvén surface would be well inside the light surface, where the field rotational velocity equals the speed of light).

Appl & Camenzind (1993a,b; hereafter ACa, ACb) investigated the asymptotic trans-field equation in the case of constant field rotation. They were first to find a non-linear analytical solution for a cylindrically collimated asymptotic field distribution (ACb). They also derived relations between the interesting jet parameters jet radius, current strength, and the field and current distribution.

In previous papers these results where used as a boundary condition for the calculation of global *two*-dimensional jet magnetospheres (Fendt et al. 1995; Fendt 1996). As it was shown, the critical solution of the wind equation along the calculated field structure asymptotically approaches the analytical force-free result (Fendt & Camenzind 1996).

However, since jet motion is connected to an accretion disk, and since the accretion disk rotates differentially, the jet magnetosphere, if it is anchored in the disk, essentially obeys differential rotation. This feature should therefore be a natural ingredient for any magnetic jet structure. How differential rotation effects the asymptotic jet equilibrium, is not obvious, since it involves collimating and de-collimating terms in the force-balance equation. Ferreira (1997) showed that differential rotation plays a major role in recollimation of jets and their asymptotic behaviour.

As a principal problem for differentially rotating relativistic jet magnetospheres, the position and shape of the singular light surface is not known *a priori*, but have to be calculated iteratively in a non-trivial way together with the flux distribution.

A differentially rotating field distribution is further interesting near the jet boundary. Here, models with a rigid field rotation imply a sharp cut off of the field rotation in the jet and in the surrounding interstellar medium, while with a differentially rotating field a smoother transition is possible.

The structure of the paper is as follows. In Sect. 2 we recall some basic equations of the theory of relativistic magnetospheres and discuss several difficulties with the solution of the Grad-Schlüter-Shafranov (hereafter GSS) equation. We evaluate the GSS equation for asymptotic cylindrical jets, including differential rotation. In Sect. 3 we discuss our results. We investigate, whether current free cylindrical jets are possible. We solve the asymptotic GSS equation for different assumptions for the field rotation and finally present a general analytic relation between the current distribution and the rotation law.

2. Structure of magnetic jets

Throughout the paper we apply the following basic assumptions: *axisymmetry*, *stationarity*, and *ideal MHD*. We use cylindrical coordinates (R, ϕ, Z) or, if normalised, (x, ϕ, z). The notation is similar to that of Fendt et al (1995) and ACa,b.

We emphasise that the term *'asymptotic'* always denotes the limit $R << Z$ and that jets with *finite* radius, $Z \to \infty$, $R < \infty$ are considered.

2.1. The force-free cross–field force–balance

With the assumption of axisymmetry, a magnetic flux function Ψ can be defined,

$$\Psi = \frac{1}{2\pi}\int \boldsymbol{B}_{\rm P}\cdot d\boldsymbol{A}, \qquad R\boldsymbol{B}_{\rm P} = \nabla\Psi \wedge \boldsymbol{e}_\phi, \tag{1}$$

measuring the magnetic flux through a surface element with radius R, threaded by the poloidal component (index 'P') of the magnetic field $\boldsymbol{B}$. With Eq. (1) the toroidal component of Ampère's law leads to the GSS equation

$$R\nabla\cdot\left(\frac{1}{R^2}\nabla\Psi\right) = -\frac{4\pi}{c} j_\phi\,, \tag{2}$$

with the toroidal component (index ϕ) of the current density $\boldsymbol{j}$. The poloidal current, defined similarly to the magnetic flux function,

$$I = \int \boldsymbol{j}_{\rm P}\cdot d\boldsymbol{A} = -\frac{c}{2} R B_\phi, \tag{3}$$

flows within the flux surfaces, $I = I(\Psi)$. The projection of the force-free, relativistic equation of motion (where inertial effects of the plasma are neglected),

$$0 = \rho_{\rm e}\,\boldsymbol{E} + \frac{1}{c}\,\boldsymbol{j}\wedge\boldsymbol{B}\,, \tag{4}$$

(with the electric field $\boldsymbol{E}$ and the charge density $\rho_{\rm e}$) perpendicular to the magnetic flux surface provides the toroidal current density,

$$\frac{1}{c} j_\phi \left(1 - \left(\frac{R\Omega_{\rm F}}{c}\right)^2\right) = \frac{1}{4\pi R}\frac{4}{c^2}\frac{1}{2}\frac{dI^2}{d\Psi} - \frac{\Omega_{\rm F}}{4\pi c^2 R}(\nabla\Psi\cdot\nabla)(R^2\Omega_{\rm F})\,. \tag{5}$$

$\Omega_{\rm F}$ is the angular velocity of the field lines and is conserved along the flux surfaces, $\Omega_{\rm F} = \Omega_{\rm F}(\Psi)$. Both the current distribution $I(\Psi)$ and the rotation law of the field, $\Omega_{\rm F}(\Psi)$, determine the source term for the GSS equation and govern the structure of the magnetosphere. Combining Eqs. (5) and (2) the cross-field force-balance can eventually be written as

$$R\nabla\cdot\left(\frac{1-(R\Omega_{\rm F}(\Psi)/c)^2}{R^2}\nabla\Psi\right) = -\frac{4}{c^2}\frac{1}{R}\frac{1}{2}\frac{d}{d\Psi}I^2(\Psi) - R\,|\nabla\Psi|^2\frac{1}{2}\frac{d}{d\Psi}\Omega_{\rm F}^2(\Psi), \tag{6}$$

which is called the modified relativistic GSS equation.

At the light surface with $R = R_{\rm L} \equiv (c/\Omega_{\rm F}(\Psi))$ the rotational velocity of the field lines equals the speed of light. Here, the GSS equation becomes singular. For differentially rotating magnetospheres the shape of this surface is not known a priori and has to be calculated in an iterative way together with the 2D solution of the GSS equation. For constant field rotation the light surface is of cylindrical shape. We choose the following normalisation,

$$\begin{aligned} R, Z &\Leftrightarrow x\,R_0, z\,R_0, \\ \Omega_{\rm F} &\Leftrightarrow \Omega_{\rm F}\,(c/R_0)\,, \\ \Psi &\Leftrightarrow \Psi\,\Psi_{\rm max}\,, \\ I &\Leftrightarrow I\,I_{\rm max}\,, \\ B_{\rm P}^2 &\Leftrightarrow y\,(8\pi\Psi_{\rm max}^2/R_0^4)\,. \end{aligned}$$

For the length scale R_0 the radius of the *asymptotic* light cylinder (see below) is selected. In order to allow for an immediate comparison to rigidly rotating magnetospheres, the normalisation is chosen such that $\Omega_{\rm F} = 1$ at $x = 1$.

With the normalisation applied, Eq. (6) can be written dimensionless,

$$x\nabla\cdot\left(\frac{1-x^2\Omega_{\rm F}^2(\Psi)}{x^2}\nabla\Psi\right) = -\; g\frac{1}{x}\frac{1}{2}\frac{d}{d\Psi}I^2(\Psi) -\; x|\nabla\Psi|^2\frac{1}{2}\frac{d}{d\Psi}\Omega_{\rm F}^2(\Psi). \qquad (7)$$

g is a coupling constant describing the strength of the current term in the GSS equation,

$$g = \frac{4I_{\rm max}^2R_0^2}{c^2\Psi_{\rm max}^2} = 4\,\left(\frac{I_{\rm max}}{10^{18}{\rm A}}\right)^2\left(\frac{R_0}{10^{16}{\rm cm}}\right)^2\left(\frac{\Psi_{\rm max}}{10^{33}\,{\rm Gcm}^2}\right)^{-2}$$

in the case of AGN, and

$$g = 4\,\left(\frac{I_{\rm max}}{10^{12}{\rm A}}\right)^2\left(\frac{R_0}{10^{14}{\rm cm}}\right)^2\left(\frac{\Psi_{\rm max}}{10^{25}\,{\rm Gcm}^2}\right)^{-2}$$

for protostellar parameters. Note that g in this paper is in accordance with the definitions in Fendt et al. (1995) and differs from the definition in ACa,b by a factor of two, $g_{\rm Fendt} = 2\,g_{\rm AC}$. A coupling constant, defined in a similar way for the differential rotation term, would be equal to unity, indicating on the important role of this effect.

2.2. Where is the asymptotic light cylinder located?

We define the *asymptotic* light cylinder, R_0, as the asymptotic branch of the light surface $R_{\rm L}(\Psi)$. Asymptotically, this quantity plays the same role for the GSS equation as the light cylinder does in the case of a rigid rotation of the magnetosphere.

All asymptotic flux surfaces within R_0 rotate slower than the speed of light and $R(\Psi) < R_{\rm L}(\Psi)$. Flux surfaces outside R_0 may rotate faster than the speed of light, here $R(\Psi) > R_{\rm L}(\Psi)$. Despite a possible degeneration of the GSS equation for a special rotation law (see below), there is only a single physical asymptotic light cylinder possible. Therefore, $R_0 \equiv R(\Psi_o) \equiv R_{\rm L}(\Psi_o)$.

It should be noted that the introduction of a light cylinder $R_{\rm L}(\Psi) = c/\Omega_{\rm F}(\Psi)$ also relies on the Ideal MHD assumption. For a non-infinite plasma conductivity a *conserved angular velocity of the field lines* $\Omega_{\rm F}(\Psi)$ cannot be defined. However, even in this case, the field may move with relativistic speed. The mathematical formalism, of course, becomes more complicated and its solution is beyond the scope of this paper. An estimate of diffusion and dynamical times scales for protostellar jets, respectively, leads to the conclusion that the Ideal MHD assumption may be appropriate (Fendt 1994). For AGN this assumption would be even more valid.

2.2.1. Stellar magnetosphere

In the case of a constant field rotation the light cylinder radius just follows from the rotational velocity of the field (and does *not* depend on the flux *distribution* $\Psi(R, Z)$). Under the assumption that the field is anchored in the stellar surface, the field rotation follows from the stellar rotational period $P_\star$. The rotational period of many protostellar jet sources is not known, but in the case of T Tauri stars it is of the order of days. Thus, we estimate the light cylinder radius

$$R_{\rm L} = 2\,10^{15}{\rm cm}\left(\frac{P_\star}{5^d}\right) = 1.4\,10^4 R_\star\left(\frac{R_\star}{2{\rm R}_\odot}\right)^{-1}\left(\frac{P_\star}{5^d}\right)\,.$$

This radius is of the order of the observationally resolved asymptotic jet radius of about 10^{15} cm (Mundt et al. 1990; Ray et al. 1996). HST observations indicate on slightly smaller jet radii of 20 AU (Kepner et al 1993).

For neutron stars the light cylinder is at

$$R_{\rm L} = 5\,10^9{\rm cm}\left(\frac{P_\star}{1^s}\right) = 4630 R_\star\left(\frac{R_\star}{10^6{\rm cm}}\right)^{-1}\left(\frac{P_\star}{1^s}\right)\,.$$

2.2.2. Disk magnetosphere

For disk magnetospheres the rotation law is determined by the flux distribution along the disk surface together with the disk rotation. If the foot point of a flux surface on the disk (here the term *foot point* denotes the position along the field line, where ideal MHD sets in) at a radius $R_{\rm D}(\Psi)$ rotates with Keplerian speed, the flux surfaces intersect the light surface at the radius

$$R_{\rm L}(\Psi) = 570 R_\star\left(\frac{R_{\rm D}(\Psi)}{R_\star}\right)^{\frac{3}{2}}\left(\frac{M}{3\,{\rm M}_\odot}\right)^{-\frac{1}{2}}\left(\frac{R_\star}{2{\rm R}_\odot}\right)^{\frac{5}{2}}$$

(here for protostellar parameters) with the mass of the central object M. The ratio between the position radius of the light surface and the light cylinder for rigid rotation is then

$$\frac{R_{\rm L}(\Psi)}{R_{\rm L}} = 0.05\left(\frac{R_{\rm D}(\Psi)}{R_\star}\right)^{\frac{3}{2}}\left(\frac{R_\star}{2{\rm R}_\odot}\right)^{\frac{3}{2}}\left(\frac{M}{3\,{\rm M}_\odot}\right)^{-\frac{1}{2}}\left(\frac{P_\star}{5^d}\right)^{-1}$$

(again for a protostellar disk magnetosphere). Is the central object a neutron star, this ratio decreases by a factor of about 100. For AGN we can estimate

$$R_{\rm L}(\Psi) = 4\,10^{15}{\rm cm}\left(\frac{R_{\rm D}(\Psi)}{R_{\rm S}}\right)^{\frac{3}{2}}\left(\frac{M}{10^{10}\,{\rm M}_{\odot}}\right)\,,$$

and in general

$$\frac{R_{\rm L}(\Psi)}{R_{\rm S}} = \sqrt{2}\left(\frac{R_{\rm D}(\Psi)}{R_{\rm S}}\right)^{\frac{3}{2}}, \qquad (8)$$

where $R_{\rm S}$ is the Schwarzschild radius of the black hole.

The question, whether or not a relativistic description is required for the jet magnetosphere, depends on the *asymptotic* radius of the flux surface, R_{∞}. If for any flux surface $R_{\rm L}(\Psi) \lesssim R_{\infty}(\Psi)$, a relativistic description of the magnetosphere is required. In the contrary, if for all flux surfaces $R_{\rm L}(\Psi) > R_{\infty}(\Psi)$, the Newtonian description is appropriate. Note that even then, for two arbitrary flux surfaces Ψ_1 and Ψ_2 with $R_{\infty}(\Psi_1) < R_{\infty}(\Psi_2)$, $R_{\rm L}(\Psi_1) < R_{\infty}(\Psi_2)$ is possible.

In principal, the asymptotic field distribution is a result of the two-dimensional force-balance of the jet, and therefore should follow from the solution of the two-dimensional GSS equation. We hypothesise that the asymptotic force-free solution will uniquely be determined by the disk flux and current distribution (and vice versa).

Our results for differentially rotating jets can hardly deliver a statement about the *absolute* value of the asymptotic jet radius, but only in terms of the asymptotic light cylinder jet radius R_0.

2.3. The asymptotic force-balance

In the asymptotic regime of a highly collimated jet structure we reduce Eq. (7) to a one-dimensional equation, equivalent to the assumption $\partial_x >> \partial_z$.

Then, $\Psi(x,z) \rightarrow \Psi(x)$, and the conserved quantities $I(\Psi)$ and $\Omega_{\rm F}(\Psi)$ can be expressed as functions of x. If we further assume a monotonous flux distribution $\Psi(x)$, the derivatives $(\partial/\partial\Psi) \rightarrow (d\Psi/dx)^{-1}(d/dx)$. Note that this excludes hypothetical solutions with a return current from our treatment (see also Sect. 3.4.1).

With the assumptions made above, the GSS Eq. (7) reduces to an ordinary differential equation of *first order* in the derivative $(d\Psi/dx)^2$,

$$\left(1 - x^2\Omega_{\rm F}^2\right)\frac{d}{dx}(\frac{d\Psi}{dx})^2 + \left(\frac{4}{x} - 2x\Omega_{\rm F}^2 - x^2\frac{d\Omega_{\rm F}^2}{dx}\right)(\frac{d\Psi}{dx})^2 + g\frac{dI^2}{dx} = 0 \qquad (9)$$

Since $(x^{-2}d\Psi/dx)^2$ is related to the magnetic pressure of the poloidal field, Eq. (9) can be rewritten as

$$\left(1 - x^2\Omega_{\rm F}^2\right)\frac{dy}{dx} - 4xy\left(\Omega_{\rm F}^2 + \frac{x}{4}\frac{d\Omega_{\rm F}^2}{dx}\right) = -\frac{g}{8\pi x^2}\frac{dI^2}{dx}. \qquad (10)$$

The magnetic flux function then follows from integration of

$$\frac{d\Psi(x)}{dx} = x\sqrt{8\pi y(x)} \qquad (11)$$

with $\Psi(x=0) = 0$. At the singular point $x = 1$ the solution $y(x)$ must satisfy the regularity condition

$$y(1) = \frac{g}{8\pi}\frac{dI^2(1)}{dx}\left(4 + \frac{d\Omega_{\rm F}^2(1)}{dx}\right)^{-1}. \qquad (12)$$

We mention that Eq. (10) can also be derived from the equation for the asymptotic force-equilibrium perpendicular to the flux surfaces,

$$\left(1 - \frac{R^2}{R_{\rm L}^2}\right)\nabla_{\perp}\frac{B_{\rm P}^2}{8\pi} - \frac{RB_{\rm P}^2}{2\pi R_{\rm L}^2}\nabla_{\perp}R - \frac{B_{\rm P}^2\Omega_{\rm F}}{4\pi c^2}\nabla_{\perp}(R^2\Omega_{\rm F}) + \frac{1}{8\pi R^2}\nabla_{\perp}(RB_{\phi})^2 = 0\,, \qquad (13)$$

where $\nabla_{\perp}$ indicates the gradient perpendicular to the flux surfaces, and where poloidal field curvature and the centrifugal force are neglected (Chiueh, Li & Begelman, 1991; ACa).

2.4. Discussion of the force-free assumption

One may question the assumption of a force-free asymptotic jet. Indeed, in a self-consistent picture of jet formation, the asymptotic jet is located beyond the collimating, non force-free wind region and beyond the fast magnetosonic surface. The asymptotic jet parameters are determined by the critical wind motion and thus, the poloidal current and the angular velocity of the field are *not* functions free of choice.

The essential point here is the assumption of a *cylindrical* shape of the asymptotic jet, an assumption, however, which is clearly indicated by the observations. The general, *non force-free* expression for the poloidal current is

$$RB_{\phi} = -4\pi\frac{\eta(\Psi)E(\Psi)}{\Omega_{\rm F}(\Psi)}\frac{x_A^2 - x^2}{1 - M^2 - x^2},$$

where $\eta(\Psi)$ is the particle flow rate per flux surface, $E(\Psi)$ is the conserved total energy, M the Alfvén Mach number, and $x_A(\Psi)$ the Alfvén radius of the flux surface. For cylindrical flux surfaces, *all* quantities on the r.h.s. are functions of Ψ, and thus, also RB_{ϕ} is a function of Ψ. Although RB_{ϕ} is not equal to the force-free current, it enters Eqs. (10) and (13) in a similar way.

The centrifugal term, which was neglected in Eq. (13), is $-\gamma^2\rho R\Omega^2\boldsymbol{e}_R$, with the plasma density ρ and plasma angular velocity Ω (see ACa). This term may be important for small plasma densities ρ, where $R\Omega$ might be large, as well as for high densities, where the toroidal plasma velocity is supposed to be small. We can estimate the importance of this term by normalising and introducing a coupling constant

$$g_M = \frac{\dot{M}_{\rm jet}cR_0^2}{\pi\Psi_{\rm max}^2},$$

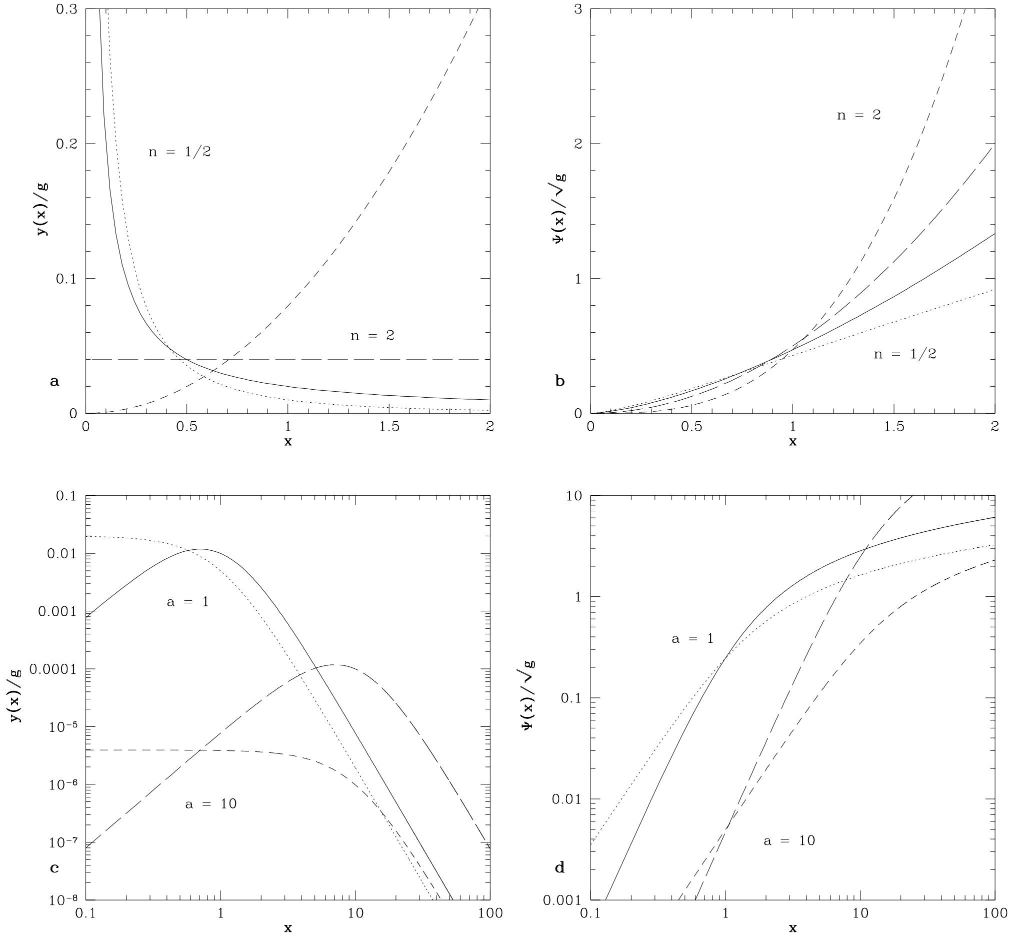

Fig. 1. a,c Magnetic pressure distribution $y(x)$ and **b,d** flux distribution $\Psi(x)$ for a field rotation law $\Omega_{\rm F}(x) = (1/x)$ (solid, long-dashed) and $\Omega_{\rm F}(x) \equiv 1$ (dotted, short-dashed). **a,b** Current distribution $I(x) = x^n$, $n = 0.5, 2$, **c,d** current distribution (22), $a = 1, 10$, $n = 2$.

which is of the order of one tenth for a jet mass loss rate $\dot{M}_{\rm jet} \simeq 10^{-2} {\rm M}_{\odot} {\rm yr}^{-1}$ and other parameters typical for AGN.

For protostellar jet parameters and a mass loss rate $\dot{M}_{\rm jet} \simeq 10^{-10} {\rm M}_{\odot} {\rm yr}^{-1}$, g_M increases by a factor of 1000. However, in this case we may expect that the Alfvén surface of the plasma motion is located well inside the light cylinder. Thus, the plasma, rotating with constant angular momentum beyond the Alfvén surface, has a decreasing and low angular velocity Ω (which is normalised to the $\Omega_{\rm F}$). The centrifugal term $\sim \Omega^2$ may become comparatively small. We emphasise that the latter arguments are rather (simplifying) assumptions than keen conclusions, as long as the true non force-free jet equilibrium is not investigated.

Contopoulos & Lovelace (1994) and Ferreira (1997) constructed *self-similar* solutions including centrifugal forces showing that the magnetic terms indeed may dominate the centrifugal term for large radii leading to a recollimation of the outflow.

2.5. Solution of the asymptotic GSS equation

Eq. (10) can be solved by the method of the variation of constants. The integrating factor of the differential equation is

$$M(x) = \exp\left(\int -4x \frac{\Omega_{\rm F}^2(x) - \frac{x}{4}\frac{d}{dx}\Omega_{\rm F}^2(x)}{1 - x^2\Omega_{\rm F}^2(x)} dx\right), \tag{14}$$

with the formal solution

$$y(x) = \frac{1}{M(x)}\left(C - \int \frac{M(x)}{1 - x^2\Omega_{\rm F}^2(x)} \frac{g\frac{d}{dx}I^2(x)}{8\pi x^2} dx\right) \tag{15}$$

Using Eq. (12), the general solution can be evaluated,

$$y(x) = \frac{1}{M(x)}\frac{g}{8\pi}\int_x^1 \frac{1}{\tilde{x}^2}\frac{M(\tilde{x})}{1 - \tilde{x}^2\Omega_{\rm F}^2(\tilde{x})}\frac{d}{d\tilde{x}}I^2(\tilde{x})d\tilde{x}\,. \tag{16}$$

As already mentioned by ACa, the solution $y(x)$ is determined by the regularity condition (12). The magnetic flux function is

$$\Psi(x) = \int_0^x \tilde{x}\sqrt{8\pi y(\tilde{x})}d\tilde{x}\,. \tag{17}$$

In the case of a constant field rotation, ACb found an analytical, non-linear solution to the asymptotic GSS equation, the flux distribution

$$\Psi(x) = \frac{1}{b}\ln\left(1 + \left(\frac{x}{a}\right)^2\right), \tag{18}$$

together with the current distribution

$$I(\Psi) = \frac{1 - e^{-b\Psi}}{1 - e^{-b}}, \tag{19}$$

leading to a certain relationship between the current distribution parameter b, the core radius a, the coupling constant g, and the asymptotic jet radius $x_{\rm jet} \equiv x(\Psi = 1)$.

3. Results and discussion

We now discuss different solutions of the asymptotic GSS equation including differential rotation. We first consider the case of vanishing poloidal current. We give an analytical solution for a special rotation law leading to a 'degeneration' of the asymptotic light cylinder. Then, Eq. (10) is solved numerically for different assumptions for the asymptotic field rotation $\Omega_{\rm F}(x)$. Finally, using a general ansatz for the asymptotic field distribution we derive a relation between $I(\Psi)$ and $\Omega_{\rm F}(\Psi)$.

In general, the differential equation for the field pressure (10) can be rewritten as a differential equation for the angular velocity of the field lines,

$$\frac{d\Omega_{\rm F}^2}{dx} + \left(\frac{4}{x} + \frac{d(\ln y)}{dx}\right)\Omega_{\rm F}^2 = \frac{g}{8\pi}\frac{1}{x^4 y}\frac{dI^2}{dx} + \frac{1}{x^2}\frac{d(\ln y)}{dx}, \qquad (20)$$

with the formal solution

$$\Omega_{\rm F}^2(x) = \frac{1}{x^4 y}\left(C + \frac{g}{8\pi}I^2(x) + \int x^2\frac{dy}{dx}\right) . \qquad (21)$$

For *physical* reasons, $\Omega_{\rm F}^2(x)$ should be monotonous (since coupled to the disk rotation), and positive for all x. In order to be consistent with the chosen normalisation, we further require $\Omega_{\rm F}^2(1) = 1$, and $x^2\Omega_{\rm F}^2 < 1$ for $x < 1$. From the latter condition, it follows that the integration constant must vanish, $C = 0$. Otherwise the rotational velocity $x\Omega_{\rm F}$ of the field would diverge for $x \rightarrow 0$. Note that, although the angular velocity may diverge with $\Omega_{\rm F} \sim 1/x^m$, $0 < m \leq 1$, the rotational velocity remains finite for $x \rightarrow 0$.

We can further see that for particular choice, a bounded current distribution with the core radius a,

$$I(x) = \frac{(x/a)^n}{1 + (x/a)^n} , \qquad (22)$$

and for $n \geq 2$ the current term in Eq. (21) does not diverge in the limit $x \rightarrow 0$, leading to finite angular field rotation (since $y(0)$ must be finite), while for $n \gtrsim 1/2$ the angular velocity diverges but not the rotational velocity, $x\Omega_{\rm F} \rightarrow$ finite value.

3.1. The case of constant or vanishing current

Now we take a look at the case of a vanishing poloidal current. A constant current, $I(x)$ = const, would imply a divergence in the field rotation.

If $I(x) = 0$, from the regularity condition (12) it follows that $y(1) = 0$. From Eq. (21) we conclude that a physical rotation law (which does not diverge at $x = 1$) requires that the numerator $\int x^2(dy/dx)dx$ vanishes together with the denominator $x^4 y$. This, however, is in contradiction with the requirement of a decreasing, *monotonous* rotation law, as it can be derived from the following. A vanishing integral $\int x^2(dy/dx)dx$ requires that the integrand changes sign at a certain position. Thus, $y(x)$ has to have a maximum (a minimum is ruled out, since $y(1) = 0$), and also the term $x^4 y$. On the other hand, the integral has a maximum too, but not necessarily at the same position. This implies that the ratio of numerator and denominator passes a point of inflection, where both terms equal, and therefore $\Omega_{\rm F}^2 = 1$. Since also $\Omega_{\rm F}(x = 1) = 1$ by definition, this is in contradiction with a monotonous rotation law.

We conclude only from asymptotic considerations that cylindrically collimated differentially rotating jets always carry a non-constant, net poloidal current. This is in agreement with previous results (Heyvaerts & Norman 1989, Chiueh et al. 1991).

3.2. A solution with degenerate light cylinder

The next case we will investigate is for a rotation law

$$\Omega_{\rm F}(x) = \frac{1}{x} . \qquad (23)$$

Now all asymptotic field lines rotate with the speed of light, and the light cylinder degenerates. Note that this does not contradict with our choice of normalisation. The length scale is measured in units of R_0, which is the light cylinder of a rigidly rotating magnetosphere. Here, $\Omega_{\rm F}(1) \equiv (\Omega_{\rm F})_{\rm rigid} = 1$.

The rotation law (23) and the corresponding field distribution may be considered as a somehow 'limiting case' for a physical field rotation. For a rotation law with a steeper slope (e.g. for $\Omega_{\rm F}^2 \sim x^{-3}$) the rotational velocity will diverge if $x \rightarrow 0$. Also, the surface $x = 1$ then plays the role of a somehow 'inverted' light cylinder since all field lines within (outside) the light cylinder rotate faster (slower) than the speed of light. Whether this behaviour could be considered as appropriate for astrophysical application also depends on the 2D field distribution.

Since for assumption (23) the derivative term of y disappears in Eq. (10), we can immediately write down the solution

$$y(x) = \frac{g}{16\pi x}\frac{d}{dx}I^2(x) , \quad \Psi(x) = \int_0^x\left(\frac{g\tilde{x}}{2}\frac{d}{d\tilde{x}}I^2(\tilde{x})\right)^{\frac{1}{2}} d\tilde{x}. \qquad (24)$$

With a current distribution $I(x) = x^n$ the field distribution is

$$y(x) = \frac{g}{16\pi}2nx^{2n-2} , \quad \Psi(x) = \frac{\sqrt{ng}}{n+1}x^{n+1} . \qquad (25)$$

This gives a rotation law for the flux surfaces

$$\Omega_{\rm F}(\Psi) = \left(\frac{\sqrt{ng}}{n+1}\frac{1}{\Psi}\right)^{1/(n+1)} \qquad (26)$$

We show the solution with bounded current distribution (22) and $n = 2$ in the Appendix. Fig. 1 displays both results in comparison with a field distribution resulting from a rigid rotation law, $\Omega_{\rm F} \equiv 1$.

We note that Contopoulos (1994) applied a similar rotation law for self-similar solutions of the 2D GSS equation, which take self-consistently into account also plasma inertia effects. With a current distribution $I(x) \sim x^{n-1}$, Eq. (24) reveals

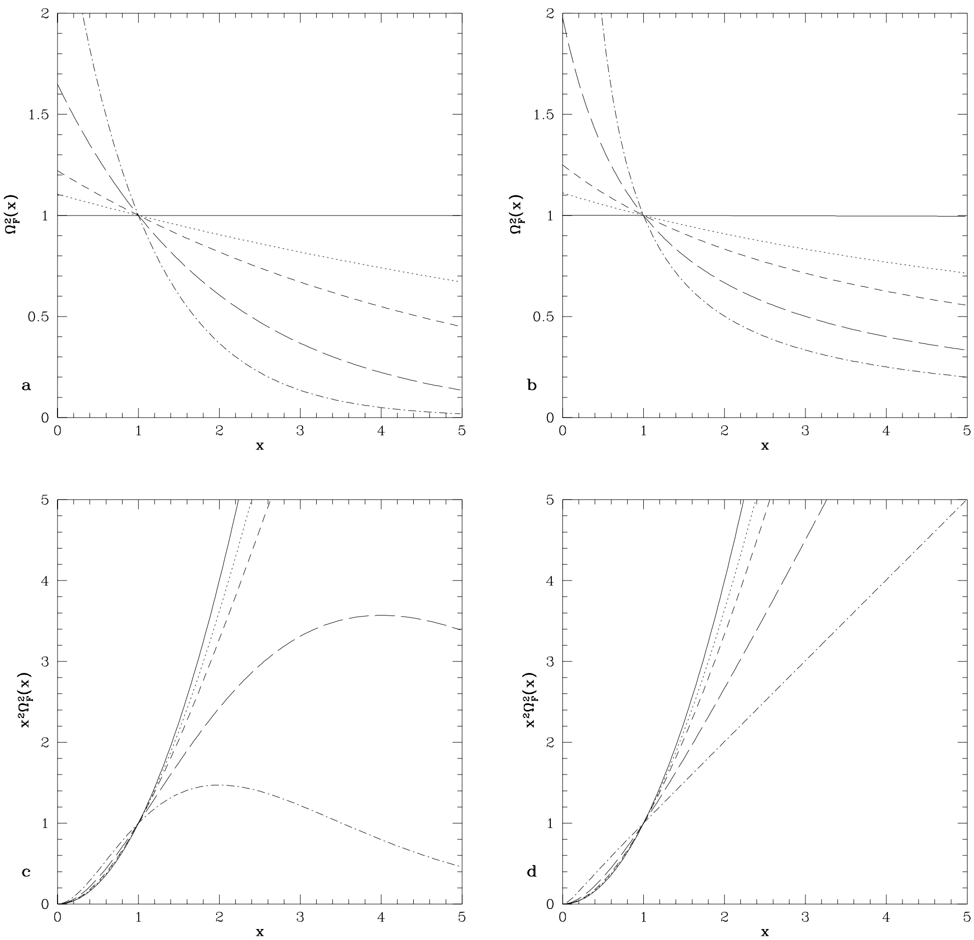

Fig. 2a–d. The rotation laws applied for the asymptotic jet magnetosphere (29) (**a**), (**c**), and (30) (**b**), (**d**). Square of the **a**, **b** angular velocity $\Omega_{\rm F}^2$, and **c**, **d** the rotational velocity $x^2\Omega_{\rm F}^2$. Parameters: $h = 0$ (solid), $h = 0.1$ (dotted), $h = 0.2$ (short-dashed), $h = 0.5$ (long-dashed), $h = 1.0$ (dotted-dashed).

$\Psi(x) \sim x^n$, which is identical to the results of Contopoulos (1994). In the force free limit, his function $H(\Psi)$ is identical to our poloidal current $(2/c)I(\Psi)$.

As a simple application of this differentially rotating field distribution, the asymptotic solutions (25) and (26) are connected to an accretion disk with Keplerian rotation, $\Omega_{\rm K}(x) = \sqrt{GM/c^2R_0}\,x^{-3/2}$ (we assume here that the flux surfaces originating in the disk rotate with this velocity).

Since the field rotation near the disk $\Omega_{\rm F}((\Psi(x))_{\rm disk}) \equiv \Omega_{\rm K}(x)$ must be the same as in the asymptotic regime, $\Omega_{\rm F}(\Psi)$, the flux distribution near the disk can be calculated,

$$(\Psi(x))_{\rm disk} = \frac{\sqrt{ng}}{n+1}\left(\sqrt{\frac{c^2R_0}{GM}}\right)^{n+1} x^{3(n+1)/2}\,. \qquad (27)$$

From the comparison of the disk flux distribution with the asymptotic flux distribution, it follows that for a certain flux surface the ratio between it's asymptotic radius, $x_{\infty,\Psi}$ and the radius near the disk $x_{D,\Psi}$ is

$$\frac{x_{\infty,\Psi}}{x_{D,\Psi}} = \sqrt{\frac{c^2R_0}{GM}\,x_{D,\Psi}}\,. \qquad (28)$$

We can further calculate the foot point of the outermost flux surface, $\Psi = 1$, from Eq. (27), and with that and Eq. (28) the 'total expansion rate' of the jet

$$\frac{x_{\rm jet}}{x_{D,\Psi=1}} = \left(\frac{(n+1)}{\sqrt{ng}}\right)^{1/3(n+1)}\left(\frac{c^2R_0}{GM}\right)^{1/3}\,.$$

The first term in this equation varies rather weakly with g, and is of the order of unity (unless g is not much larger or much less than unity). For the second term we calculate for AGN ($M = 10^{10}\,{\rm M}_\odot$, $R_0 = 10^{16}\,$cm) a number value of about 2, which is surprisingly small, and for protostars ($M = 3\,{\rm M}_\odot$, $R_0 = 10^{15}\,$cm) a value of $\sim$ 1200, respectively. This result may indicate on an intrinsic difference between the two jet sources. However, we should keep in mind that inertial forces may change the protostellar jet expansion rate and that the assumed current distribution might not be appropriate.

Comparing the field distribution near the disk (27) and in the asymptotic region (25) at small radii $x < 1$,

$$\frac{\Psi(x)_{\rm disk}}{\Psi(x)_\infty} = \left(\frac{c^2R_0}{GM}x\right)^{(n+1)}\,,$$

we may principally expect a recollimation of certain flux surfaces, depending on the source parameters M, R_0 and the ra-

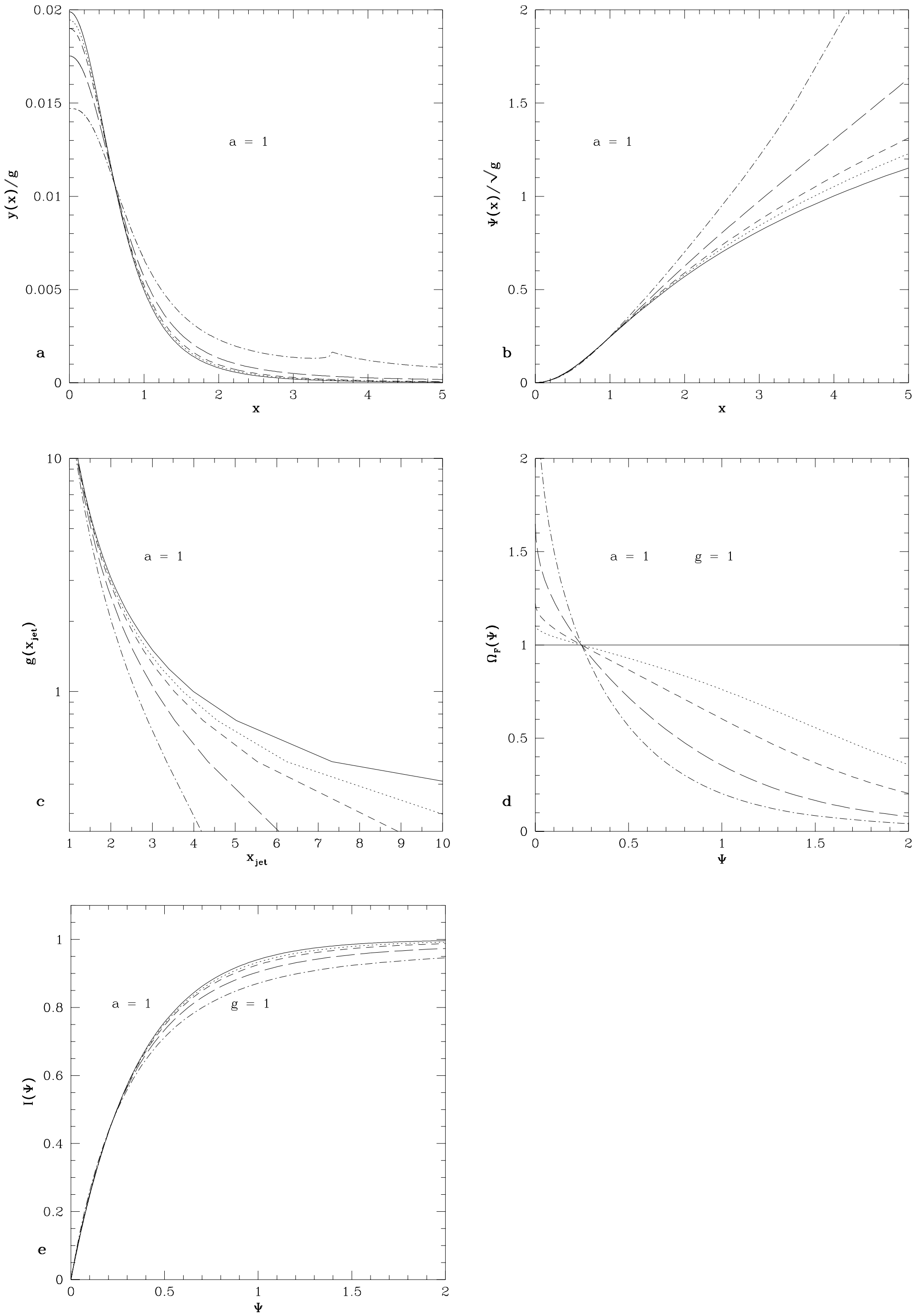

Fig. 3a–e. a Magnetic pressure distribution $y(x)$ and **b** flux distribution $\Psi(x)$. **c** Coupling constant as a function of the jet radius. The field rotation law is (29) with different steepness parameters, h = 0 (solid), h = 0.1 (dotted), h = 0.2 (short-dashed), h = 0.5 (long-dashed), h = 1.0 (dotted-dashed). The solid curves coincide with the analytical result from ACb. Note that the solid curves correspond to the dotted curves in Fig. 1c and 1d.

dius x. However, we believe that such kind of conclusions (e.g. 'recollimation predominantly for low mass AGN') might be exaggerated, since not very much is known about the disk field distribution and rotation, especially for small radii near the star, black hole, or disk boundary layer.

3.3. Numerical solutions of the asymptotic GSS including differential rotation

In this section *numerical* solutions to the asymptotic GSS equation with differential rotation are presented. Here, the current distribution is prescribed, and Eq. (10) is solved for different assumptions for the rotation law, $\Omega_{\rm F}(x) = \Omega_{\rm F}(\Psi(x))$.

In order to allow a comparison with rigid rotation solutions we chose a bounded current distribution (22) (with n = 2) in parallel to the work of ACa,b. For the rotation law we require that

(i) it is finite at $x = 0$, (ii) $\Omega_{\rm F}(x=1) = 1$, in accordance with the normalisation, and (iii) $\Omega_{\rm F}^2 > 0 \;\forall\; \Psi < 1$. These requirements are satisfied by e.g. the following functions,

$$\Omega_{\rm F}^2(x) \;=\; e^{h-hx} \tag{29}$$

$$\Omega_{\rm F}^2(x) \;=\; \frac{\exp\left(\frac{1}{x+f}\right)}{hx+1}\,,\;\; f \equiv \frac{1}{\ln(h+1)} - 1\,, \tag{30}$$

where h plays the role of a steepness parameters (see Fig. 2).

There is a further condition (iv) for a rotation law. Rotation laws remaining valid for $x \rightarrow \infty$, have to be flatter than $\Omega_{\rm F} \sim 1/x$. Otherwise the rotational *velocity* of the field lines will pass a maximum and finally decreases to values $x^2\Omega_{\rm F}^2 \leq 1$ (Fig. 2c). Note that ansatz (29) cannot be applied for arbitrarily large radii in the case of a high steepness parameter h. Since rotation law (29) is applied for a finite flux distribution, there is no serious problem as long as the turn-over of the rotational velocity is located beyond the jet radius. Ansatz (30) is more general, however, the analytical expressions look more complicated.

In Fig. 3 we display the numerical solutions of the asymptotic force balance for ansatz (29). A solution with ansatz (30) looks very similar, we therefore omitted the plot. The solid curves show the field distribution with constant field rotation coinciding with the result of ACb, the other curves the result with increasing steepness of the rotation law, respectively.

The small peak in the field pressure (Fig. 3a) along the solution with the very steep rotation law results from numerical difficulties with the above mentioned decrease of rotational velocity for large radii and does not appear for the other ansatz.

From the solutions $\Psi(x)$ and $I(x)$ or $\Omega_{\rm F}(x)$, we can derive the distribution of the conserved quantities $I(\Psi)$ and $\Omega_{\rm F}(\Psi)$ (Fig. 3d, 3e), which could be applied for force-free 2D calculations.

Fig. 3c shows the relation between the coupling constant (measuring the strength of the poloidal current) and the jet radius. In order to obtain jets with the same radius, the current strength has to be increased with increasing steepness of $\Omega_{\rm F}$. The same behaviour is mirrored in Fig. 3e, if we compare the poloidal current at the jet boundary, $I(\Psi = 1)$, for different h.

The force-equilibrium is affected by differential rotation predominantly in the outer part of the jet. The field distribution within the core radius a of the asymptotic jet is not concerned very much by differential rotation, although a slight decollimating effect can be observed. The behaviour changes beyond of $x = a$, where the collimating effect is stronger than the de-collimation effect in the inner part.

Our results clearly show that differential rotation has a *collimating* influence. Depending on the steepness parameter, the asymptotic jet radius (defined by $\Psi = 1$) varies by a factor up to 2, which could be even larger for a lower coupling g. Note that the spatial scaling is in terms of the asymptotic jet radius R_0. This parameter, however, and thus the absolute scaling can only be inferred from a 2D solution. In Sect. 2.2 we gave arguments that, due to the rapid rotation of the accretion disk, R_0 could be closer to the jet axis compared to solutions with constant rotation $\Omega_{\rm F} = \Omega_\star$.

3.4. A non-linear analytical solution

In this section we derive a general analytical solution for the rotation law $\Omega_{\rm F}(\Psi)$. We assume a form of flux distribution parameterised as in Eq. (18). However, in contrary to the case of rigid rotation, the parameter $b = \ln(1 + (x_{\rm jet}/a)^2)$ is not *a priori* coupled to the current distribution (e.g. Eq. 19). Then, the asymptotic GSS can be transformed into an ordinary differential equation for $\Omega_{\rm F}^2$,

$$\frac{d}{d\Psi}\Omega_{\rm F}^2(\Psi) + \frac{2b}{e^{b\Psi}-1}\Omega_{\rm F}^2(\Psi) \;=\; \frac{g\,b^2}{4}\frac{(e^{b\Psi})^2}{(e^{b\Psi}-1)^2}\frac{d}{d\Psi}I^2(\Psi) - \frac{1}{a^2}\frac{2\,b}{e^{b\Psi}-1} \tag{31}$$

Now we investigate, whether a combination of current distribution and rotation law can be found, which is consistent with the chosen flux distribution. The general solution of Eq. (31) is

$$\Omega_{\rm F}^2(\Psi) = \frac{C + \frac{1}{4}g\,b^2 I^2(\Psi) - a^{-2}e^{-b\Psi}(e^{-b\Psi}-2)}{(1-e^{-b\Psi})^2}\,, \tag{32}$$

with the integration constant C. This solution diverges for $\Psi \rightarrow 0$ unless $C = -1/a^2$. Thus, we obtain

$$\Omega_{\rm F}^2(\Psi) = \frac{1}{4}g\,b^2\frac{I^2(\Psi)}{(1-e^{-b\Psi})^2} - \frac{1}{a^2}\,, \tag{33}$$

and vice versa a relation for the current distribution in terms of $\Omega_{\rm F}(\Psi)$. In the limit $\Psi \rightarrow 0$ the solution approaches

$$\lim_{\Psi\rightarrow 0}\Omega_{\rm F}^2(\Psi) = \frac{1}{a^2} + \frac{g}{8}\lim_{\Psi\rightarrow 0}\frac{d^2}{d\Psi^2}I^2(\Psi)\,. \tag{34}$$

For a current distribution (19) we end up with the result of ACb with constant angular velocity of the field, $\Omega_{\rm F} = 1$.

Since by definition $I(\Psi = 1) = 1$, $x_{\rm jet} = a\sqrt{e^b - 1}$, we can derive an expression for the coupling constant

$$g = \frac{4}{b^2}(1-e^{-b})^2\left(\Omega_{\rm F}^2(1) + \frac{e^b-1}{x_{\rm jet}^2}\right)\,. \tag{35}$$

Eq. (35) is visualised in Fig. 4. We see that differential rotation plays a dominant role only for low-g / low-b jets, i.e. jets with low poloidal current and a broad field distribution (i.e. large core radius). Note that although a is shifted to lower values for steeper differential rotation, the magnetic flux $\Psi(x = a)$ remains unchanged. In the limiting case of rigid rotation the parameter b describes steepness of the poloidal current distribution. We can rewrite Eq.(35) in terms of the core radius a of the field distribution

$$g = 4\frac{\Omega_{\rm F}^2(1) + \left(1/a\right)^2}{\ln(1+(x_{\rm jet}/a)^2)}\left(\frac{\left(x_{\rm jet}/a\right)^2}{1+\left(x_{\rm jet}/a\right)^2}\right)^2 \tag{36}$$

This shows that in order to obtain the same asymptotic magnetic jet structure (with the same parameters a, b, or $x_{\rm jet}$ in Eq. (18)), the current has to be larger (parameterised by the coupling constant g) in the case of larger gradients of the rotation law. Similarly, for a fixed ratio ($x_{\rm jet}/a$) and g, but decreasing $\Omega_{\rm F}(1)$, also the core radius a (and thus $x_{\rm jet}$) is decreasing.

The thick line in Fig. 4 is the limiting value for the coupling constant g for rigid rotation, where the core radius a diverges (ACb). It corresponds to a minimum current required for rigid rotating magnetic jets, $g_\infty = 4\,(1-e^{-b})^2/b^2$. In the case of differential rotation, this value is decreased by a factor $\Omega_{\rm F}(1)^2$.

Eqs. (35) and (36) are a general result resting only on the assumption of the field distribution (18). No assumption was yet made about the *function* $\Omega_{\rm F}(\Psi)$. Any solution $I(\Psi)$, $\Omega_{\rm F}(\Psi)$ has to lie within the limiting curves of $\Omega_{\rm F}(1) = 1$ and $\Omega_{\rm F}(1) = 0$ in Fig. 4. The ratio of the coupling constants for constant rotation ($\Omega_{\rm F}(1) = 1$) and for maximum differential rotation ($\Omega_{\rm F}(1) = 0$) is

$$\frac{g_{\rm max}}{g_{\rm min}} = 1 + \frac{x_{\rm jet}^2}{e^b - 1}\,. \qquad (37)$$

Again we derive the 'total expansion rate' similar to Sect. 3.2 by comparison of the asymptotic solution with Keplerian disk rotation,

$$\frac{x_{\rm jet}^3}{x_{D,\Psi=1}^3}\left(\frac{GM}{c^2 R_0}\right) = \frac{1}{4} g b^2 \frac{x_{\rm jet}^3}{\left(1-e^{-b}\right)^2} - (e^b - 1)\,x_{\rm jet}, \qquad (38)$$

where no assumption was made about a specific field rotation. Strong currents and large asymptotic jet radii imply a strong opening of the flux surfaces. If we rewrite Eq. (38) in terms of the field rotation,

$$\frac{x_{\rm jet}}{x_{D,\Psi=1}} = \left(\frac{c^2 R_0}{GM}\right)^{\frac{1}{3}} x_{\rm jet}\,\Omega_{\rm F}^{\frac{2}{3}}(1) = \left(2\frac{R_0}{R_{\rm S}}\right)^{\frac{1}{3}} x_{\rm jet}\,\Omega_{\rm F}^{\frac{2}{3}}(1)\,, \qquad (39)$$

we see that a stronger gradient in the field rotation (a lower value of $\Omega_{\rm F}(\Psi = 1)$) leads to a lower expansion rate. A vanishing field rotation of the outermost flux surface leads to a vanishing, unphysical, expansion rate.

With reasonable numerical parameters the different central objects (see Sect. 2.2), the numerical values for the expansion rate are

$$\frac{x_{\rm jet}}{x_{D,\Psi=1}} = 2\,x_{\rm jet}\,\Omega_{\rm F}^{\frac{2}{3}}(1)\left(\frac{M}{10^{10}\,{\rm M}_\odot}\right)\left(\frac{R_0}{10^{16}{\rm cm}}\right)^{-1}$$

in the case of AGN, and

$$\frac{x_{\rm jet}}{x_{D,\Psi=1}} = 600\,x_{\rm jet}\,\Omega_{\rm F}^{\frac{2}{3}}(1)\left(\frac{M}{3\,{\rm M}_\odot}\right)\left(\frac{R_0}{10^{15}{\rm cm}}\right)^{-1}$$

for protostellar objects.

We may assume that AGN jets are highly relativistic with $1 << x_{\rm jet} \sim 100$, and therefore are strong *differential* rotators, $1 >> \Omega_{\rm F}(1) \sim 0.1$. Their expansion rate would then be of the order of 50. In the case of protostars $x_{\rm jet} \sim 1$, and thus $\Omega_{\rm F}(1) \sim 1$. The expansion rate would then be of the order of 600. The applied number values for $x_{\rm jet}$ and $\Omega_{\rm F}(1)$ are only raw estimates, indicating 'steep' or 'flat' rotation laws and 'highly' or 'weakly' relativistic field rotation, respectively.

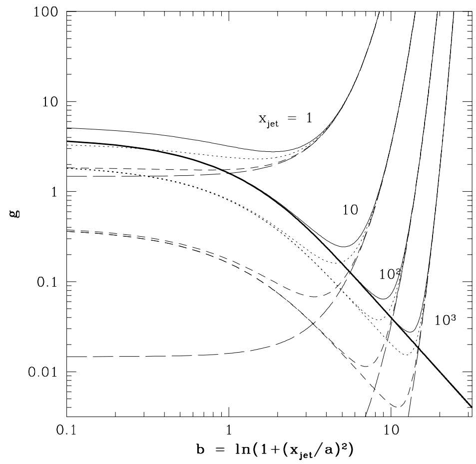

Fig. 4. Interrelation between the jet parameters g, $b = \ln(1+(x_{\rm jet}/a)^2)$, $x_{\rm jet}$ and the angular field velocity at the jet boundary, $\Omega_{\rm F}^2(\Psi = 1)$. $\Omega_{\rm F}^2(1) = 1$ (solid), $\Omega_{\rm F}^2(1) = 0.5$ (dotted), $\Omega_{\rm F}^2(1) = 0.1$ (short-dashed), $\Omega_{\rm F}^2(1) = 0$ (long-dashed). The thick solid curve is the boundary of the forbidden regime, where no rigid rotating jet solutions are possible. The solid curves coincide with the result from ACb.

Keeping all the uncertainties in mind, we may generally expect lower expansion rates for the AGN. Especially the expansion rates for protostars have to be taken with care (see also discussion end of Sect.3.2). However, a rather general conclusion might be that high mass, fast rotating AGN have higher jet expansion rates than their low mass slower rotating counterparts.

If we rewrite Eq. (35) we find an expression for the ratio of the jet radii in terms of the field rotation of the outermost flux surface.

3.4.1. The question of non-monotonous flux distribution

We note a general difficulty with non-monotonous flux distributions. In this case the jet magnetosphere would consist of flux surfaces with different foot points, but with the same absolute flux, e.g. $\Psi_1 = \Psi_2$. These flux surfaces are not directly connected within the integration domain.

There is no physical reason, why they should not carry a different poloidal current, as long it is conserved along Ψ_1 and Ψ_2, respectively. However, in this case the description of the poloidal current as a function $I(\Psi)$, seems to fail. Instead it is supposed, that always $I(\Psi_1) = I(\Psi_2)$, and one *has* to assume such kind of current distribution.

The problem is more serious for the other 'free' function, the field rotation $\Omega_{\rm F}(\Psi)$. Here, if we suppose an accretion disk as source for the magnetic flux, all foot points of the flux surfaces *must* rotate with monotonously decreasing angular velocity. Again, the description does not support a different field rotation for Ψ_1 and Ψ_2. This statement is also valid for a non force-free description.

We conclude that the monotonous disk rotation could only support monotonous flux distributions. Therefore, assumption (18) for the analytical solution seems to be rather general.

3.4.2. A special analytical rotation law

As an example for a current distribution appropriate for differential rotation we may chose

$$I(\Psi) = B\left(1 - e^{-b\Psi}\right) e^{-d\Psi}; \qquad B \equiv \left(1 - e^{-b}\right)^{-1} e^{d}. \quad (40)$$

The steepness parameter d describes the variation from constant rotation. This leads to a field rotation

$$\Omega_{\rm F}^2(\Psi) = \frac{1}{4} g\, b^2 B^2 e^{-2d\Psi} - \frac{1}{a^2}\,, \quad (41)$$

The jet radius is by definition at $\Psi = 1$, and from Eq. (18) it follows $x_{\rm jet} = a\sqrt{e^b - 1}$. Since $\Omega_{\rm F}(\Psi(x = 1)) = 1$, we calculate for the flux distribution parameter

$$a^2 = \left(\left(\frac{1}{4} g b^2 B^2 \right)^{1/(1+2d/b)} - 1 \right)^{-1}. \quad (42)$$

Again, $d = 0$ gives the result derived by ACb. Otherwise, a and also $x_{\rm jet}$ is decreased for fixed g and b.

The expression for the coupling constant is

$$g = \frac{4}{b^2} (1 - e^{-b})^2 e^{-2d} \left(1 + \frac{e^b - 1}{x_{\rm jet}^2} \right)^{1+(2d/b)}. \quad (43)$$

The angular velocity of the outermost flux surface is

$$\Omega_{\rm F}^2(\Psi = 1) = 1 + \frac{1}{4} g b^2 B^2 e^{-2d} - \left(\frac{1}{4} g b^2 B^2 \right)^{1/(1+2d/b)}. \quad (44)$$

The interrelation of the parameters g, b, d and $x_{\rm jet}$ is similar to Fig. 4. However, the parameter d has to be chosen such that $\Omega_{\rm F}^2(x_{\rm jet}) \gtrsim 0$, and $g(b; d, x_{\rm jet})$ lies within the limiting curves of $\Omega_{\rm F}(1) = 1$ and $\Omega_{\rm F}(1) = 0$ in Fig. 4.

4. Conclusions

In this paper the asymptotic force-balance across collimated magnetic flux surfaces was investigated. Relativistic effects due to rapid rotation of the field as well as differential rotation was included in the treatment.

The related astrophysical scenario is that of a highly collimated magnetic jet originating in an accretion disk, as observed in active galactic nuclei, galactic high energy sources with superluminal jets, and also protostellar jets with non-relativistic jet motion.

We presented numerical solutions of the asymptotic jet equilibrium for different assumptions of the field rotation. For a general assumption for the asymptotic field distribution we also derived an analytical solution.

The main results are the following

- *Differential rotation* always leads to a *decrease of the jet radius* in terms of the asymptotic light cylinder radius.
- This effect can be balanced by an increase of the poloidal current.
- The inner structure of the jets remains more or less unchanged, the outer part becomes 'compressed' by differential rotation.
- Jet expansion rates could be estimated under the assumption of a certain rotation law for the foot points of the field (e.g. Keplerian).
- A general analytical solution was derived for the asymptotic flux distribution together with the rotation law of the field lines and the current distribution.

Depending on the steepness of the rotation law, the ratio in the jet radius between jets with and without differential rotation can be of the order of two. We also showed that differential rotation plays a role only for jets with low poloidal current and a broad field distribution.

In order to maintain jets with the same jet radius, but with a different gradient of field rotation, the strength of the poloidal current must be increased. In this sense, differential rotation may be considered as collimating effect and poloidal currents as decollimating effect. However, compared to the rigid rotating field distribution, the minimum poloidal current required is decreased by a factor, which depends on the rotation rate of the outermost flux surface.

While within the asymptotic one-dimensional limit jets with arbitrary radius could be obtained, there are indications that 2D solutions of the relativistic GSS equation (but without differential rotation) only exist for asymptotic jet radii of the order of several light cylinder radii (Fendt et al 1995, Fendt 1996). It was impossible to obtain numerical solutions with jet radii larger than ~ 5 light cylinder radii. This result was not caused by numerical effects. The results of the present paper indicate that the jet radii are even smaller.

A central question is therefore the scaling of light cylinder radius in terms of stellar (or black hole) radii. This, however, could only be inferred from a two-dimensional solution of the trans-field equation. We believe that inclusion of inertial effects would possibly widen the jet. However, one should keep in mind that in the case of self-similar jets Contopoulos & Lovelace (1994) and Ferreira (1997) have shown that centrifugal forces could be balanced by magnetic tension leading to a recollimation of the jet.

A critical point of the present investigation is that the interaction between the jet boundary and the ambient medium is not included in the force-balance. Hence, the question whether the jet is self-collimated or pressure collimated by the ambient medium cannot be answered. However, if we take a certain jet radius as given (by e.g. observational arguments), the results of this paper give examples of the local force-free force-balance of a jet with such a radius. In this picture the field pressure at the jet boundary must be balanced by the external pressure. Smaller or larger jet radii would change the jet parameters accordingly.

By comparing the field rotation near the foot points of the field lines (near a 'disk') and in the asymptotic regime, we were able to give some estimates on the expansion rate of the jets. Protostellar jets seem to have high expansion rates of the order of 1000, but these values are biased by the force-free assumption for the force-balance. Expansion rates of AGN jets are lower, a typical value might be 10. It can be said that high-mass fast-rotating AGN jet expansion rates are expected to be higher than those from low-mass slow rotating ones.

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Appendix A: another analytical solution with degenerate light cylinder

Here we give the analytical expression for the field pressure and flux distribution for a solution with $\Omega_{\rm F}(x) = (1/x)$ and a bounded current distribution (22) with $n = 2$,

$$y(x) = \frac{g}{16\pi}\,\frac{4}{a^2}\,\frac{(x/a)^2}{\left(1+(x/a)^2\right)^3}\,,$$

$$\Psi(x) = \sqrt{2ga^2}\left(\ln\left(\frac{x}{a} + \sqrt{1+(x/a)^2}\right) - \sqrt{\frac{(x/a)^2}{1+(x/a)^2}}\right)\,.$$

The field rotation law can be expressed by an implicit equation

$$\frac{\Psi(\Omega_{\rm F})}{\sqrt{2ga^2}} = \left(\ln\left(\frac{1+\sqrt{1+(a\Omega_{\rm F})^2}}{a\Omega_{\rm F}}\right) - \sqrt{\frac{1}{1+(a\Omega_{\rm F})^2}}\right)$$

Suppose that we have Keplerian rotation of the foot points along the disk, $\Omega_{\rm F} = \Omega_{\rm K} = \sqrt{GM/c^2R_0}x^{-3/2}$, it follows for the disk flux distribution

$$(\Psi(x))_{\rm disk} = \sqrt{2ga^2}\left(\ln\left(\frac{x^{3/2}}{a\sqrt{\tilde{a}}}\left(1+\sqrt{1+\frac{a^2\tilde{a}}{x^3}}\right)\right) - \sqrt{\frac{x^3}{x^3+a^2\tilde{a}}}\right)\,, \qquad \text{(A1)}$$

where $\tilde{a} \equiv (R_{\rm S}/2R_0)$. For the 'total expansion rate' $(x_{\infty,\Psi=1}/x_{D,\Psi=1})$ of the jet we derive an implicit equation

$$\ln\left(\frac{\sqrt{\tilde{a}}x_{\infty,\Psi=1}}{x_{D,\Psi=1}^{3/2}}\,\frac{1+\sqrt{1+a^2/x_{\infty,\Psi=1}^2}}{1+\sqrt{1+a^2\tilde{a}/x_{D,\Psi=1}^3}}\right) = \\ = \left(1+\frac{a^2}{x_{\infty,\Psi=1}^2}\right)^{-\frac{1}{2}} - \left(1+\frac{a^2\tilde{a}}{x_{D,\Psi=1}^2}\right)^{-\frac{1}{2}}$$

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