Numerical Solution of Star Formation Equations

Description of protostellar collapse (Conservation equations):
Continuity equation, Momentum equation, Poisson equation, Energy equation

What are the problems: Scales, fragmentation, loss of magnetic field

(1) Density needs to increase from molecular cloud density of 10^4-10^5 cm^{-3} to 10^{24} cm^{-3} as mean solar density: 20 orders of magnitude
(2) Central temperature needs to increase from 10 K to 10^7 K in order to start fusion
(3) Specific angular momentum needs to decrease:

	J/M (in cm^2s^{-1})
Molecular clump	10^{23}
Binary (P~10^4 yr)	$4x10^{20}$-10^{21}
Binary (P~10 yr)	$4x10^{19}$- 10^{20}
Binary (P~ 3yr)	$4x10^{18}$- 10^{19}
T Tauri star	10^{17}
Sun	10^{15}
Jupiter orbit	10^{20}

What are the problems: Scales, fragmentation, loss of magnetic field

(4) Most of the stars are binaries or occur in higher-order hierarchical systems: fragmentation is needed

(5) Magnetic flux loss is necessary:

Dense core with M_c=1M_{sun}, R_c=0.07 pc, B_c=30 μG

T Tauri star: $R_1 = 5\,R_{sun}$

Flux freezing: $B\,R^2$ = const.

$B_1 = 2 \times 10^7$ G - This is much larger than what is observed for T Tauri stars

Next steps:

Formulation of all equations with the necessary material equations and boundary and initial conditions.

A few remarks

Protostellar evolution from an instable cloud core

a) Dynamical collapse ($t \approx t_{ff}$)

b) Accretion phase after formation of hydrostatic core (accretion from envelope)

Pre-main sequence evolution: Energy source from quasistatic contraction of the core

Mathematical and physical problem:

a) Dynamical problem with dynamics on different scales

b) Complex physics ---> equations are non-linear

c) Fragmentation must be treated (2D/3D)

(a) CONTINUITY EQUATION (1)

$$\frac{d\rho}{dt} + \nabla\ (\ \rho\vec{v}) = 0$$

(b) MOMENTUM EQUATIONS (3)

$$\frac{\partial(\rho v_r)}{\partial t} + \nabla(\rho v_r \vec{v}) = -\left(\rho\frac{\partial\Phi}{\partial r} + \frac{\partial p}{\partial r}\right) + \frac{\rho}{r}(v_\Theta^2 + v_\Phi^2)$$

$$\frac{\partial(\rho v_\Theta)}{\partial t} + \nabla(\rho v_\Theta \vec{v}) = -\ \frac{1}{r}\left(\rho\frac{\partial\Phi}{\partial_\Theta} + \frac{\partial p}{\partial_\Theta}\right) - \frac{\rho}{r}\ (v_r\ v_\Theta\ -\ v_\Phi^2 \cot\Theta)$$

$$\frac{\partial(\rho A)}{\partial t} + \nabla(\rho A \vec{v}) = -\ \left(\rho\frac{\partial\Phi}{\partial_\Phi} + \frac{\partial p}{\partial_\Phi}\right)$$

$A = r\ \sin\ \Theta\ v_\Phi$ (specific angular momentum)

(c) POISSON'S EQUATION (1)

$$\nabla^2\Phi = 4\ \pi\ G\ \rho$$

(e) ENERGY EQUATION (1)

$$\frac{\partial(\rho e)}{\partial t} + \nabla(\rho e\vec{v}) + p\ \nabla\vec{v} = L$$

$L = 4\ \pi\rho\kappa\ (J - B)$: time rate of change of energy per unit volume due to radiative transfer

(f) RADIATION EQUILIBRIUM (1)

$$\nabla\vec{H} + \rho\kappa\ (\ J - B\ (T)\ ) = 0$$

(g) FLUX (EDDINGTON APPROXIMATION) (3)

$$\vec{H} = -\ \frac{1}{3\kappa\rho}\nabla\ J$$

(gray radiative transfer: use of Rosseland mean opacity guarantees that radiative flux is identical to that in frequenc: dependent case)

(h) MATERIAL EQUATIONS

$e = e\,(\rho, T)$

$\kappa = \kappa(\rho, T)$ see Wuchterl (1990)

$p = p(\rho, T)$

3D $\rightarrow$ 2D equations: dropping all partial derivates with respects to ϕ

2D $\rightarrow$ 1D equations: setting terms with v_θ or v_ϕ equal to zero, neglecting the equations for v_θ and $A(v_\phi)$; dropping terms with partial derviatives with respect to θ

INITIAL CONDITIONS

STANDARD I.C.: Spherical cloud of uniform density and temperature, which initially everywhere at rest

STANDARD I.C. + SOLID BODY ROTATION:

- Isothermal rotating clouds:
  $\alpha_i = E_i^{th} \,/\, E_i^{grav}$; $\beta_i = E_i^{th} \,/\, E_i^{grav}$
- Adiabatic rotating clouds:
  γ ($p \propto \rho^\gamma$), α_i, β_i

  (3D: Form of perturbation has to be added)
- Nonisothermal clouds:
  α_i, β_i, M_i, T_i

BOUNDARY CONDITIONS (Φ)

No mass exterior to the protostar

Mathematical formulation (see Boss 1987)

13 physical quantities:

ρ (density)
$\vec{v}$ (velocity)
Φ (gravitational potential)
J (mean intensity)
$\vec{H}$ (Radiation flux)
p (gas pressure)
e (specific internal energy)
κ (opacity)
T (temperature)

13 equations:

(10 coupled, nonlinear, partial diff. equations of first and second order with strongly variable coefficients + 3 algebraic material equations)

- **Eulerian form** (equations are written with respect to a fixed frame of reference)

- **Spherical coordinates** r, θ, ϕ

Initial and boundary conditions

Standard initial conditions (Larson 1969)

Homogeneous sphere with const. T and density which is at rest at t=0

Standard initial conditions plus rigid rotation ω_i

Isothermal rotating cloud:

$\alpha_i = E_i^{th}/E_i^{grav} = 5/2\ R_i\ R\ T_i/G\ M_i\ \mu$

$\beta_i = E_i^{rot}/E_i^{grav} = \omega_i^2/4\ \pi\ G\ \varrho_i$

Adiabatic rotating cloud: $p \sim \varrho^{\gamma}$; α_i and β_i

Non-isothermal clouds: α_i and β_i , M_i, T_i

Initial and boundary conditions

Boundary conditions

Why do we need boundary conditions:

Two equations of second order for Φ and J

- Constant J or J from constant temperature condition
- Φ – no mass outside protostar (observations – not realistic)

Mathematical Formulation and Solution

Coordinate System: Lagrangian or Eulerian formulation

Eulerian formulation: Numerical diffusion has to be minimized
Lagrangian formulation: No numerical diffusion (no nonlinear advection term)

- Very complicated for multi-dimensional hydrodynamical processes
- Rezoning of grids (numerical diffusion re-introduced)

Application of particle methods instead of grid methods:

Fluid divided in cells – „particles“ which move under the action of external forces and interact

„Smoothed Particle Hydrodynamics“ (SPH)
(Lucy 1977; Gingold & Monaghan 1977)

Fluid divided in discrete elements. These „particles“ have a spatial distance („smoothing length“) over which their properties are smoothed by a kernel function

$A(r) = \Sigma_j\, m_j\, A_j/\varrho_j\, W(\,|\, r - r_j|, h)$ Here W is the kernel function.

Advantages:

- Large density gradients can be treated
- Boundaries can be easily introduced (non-spherical clouds)
- Gravitational interaction can be easily integrated.

Disadvantage:

Limited stability and accuracy in complex flows

Grid-based methods (resolution increase)

a) Adaptive grids

Number of grid points is constant; will be increased where strong gradients occur

b) Nested grids

Individual grid points will be split; number of grid points no longer constant.

Adaptive mesh refining (AMR):

Dynamical gridding during simulation; starts with coarsely resolved Cartesian grid. Then individual cells are tagged for refinement (e.g. condition – mass per cell should remain constant

GMC simulations have reached 10^{-7} effective resolution per initial radius

Solutions: Explicit and implicit methods

$\delta\, \mathbf{u}/\delta t = L\, \mathbf{u}$; $\mathbf{u} = \mathbf{u}\,(\mathbf{r}, t)$ (L – non-linear operator)

$\mathbf{u}^{n+1} = \mathbf{u}^n + L\,\mathbf{u}\,(1 - \varepsilon)\,\Delta t + L\,\mathbf{u}^{n+1}\,\varepsilon\,\Delta t$; ε – interpolation parameter

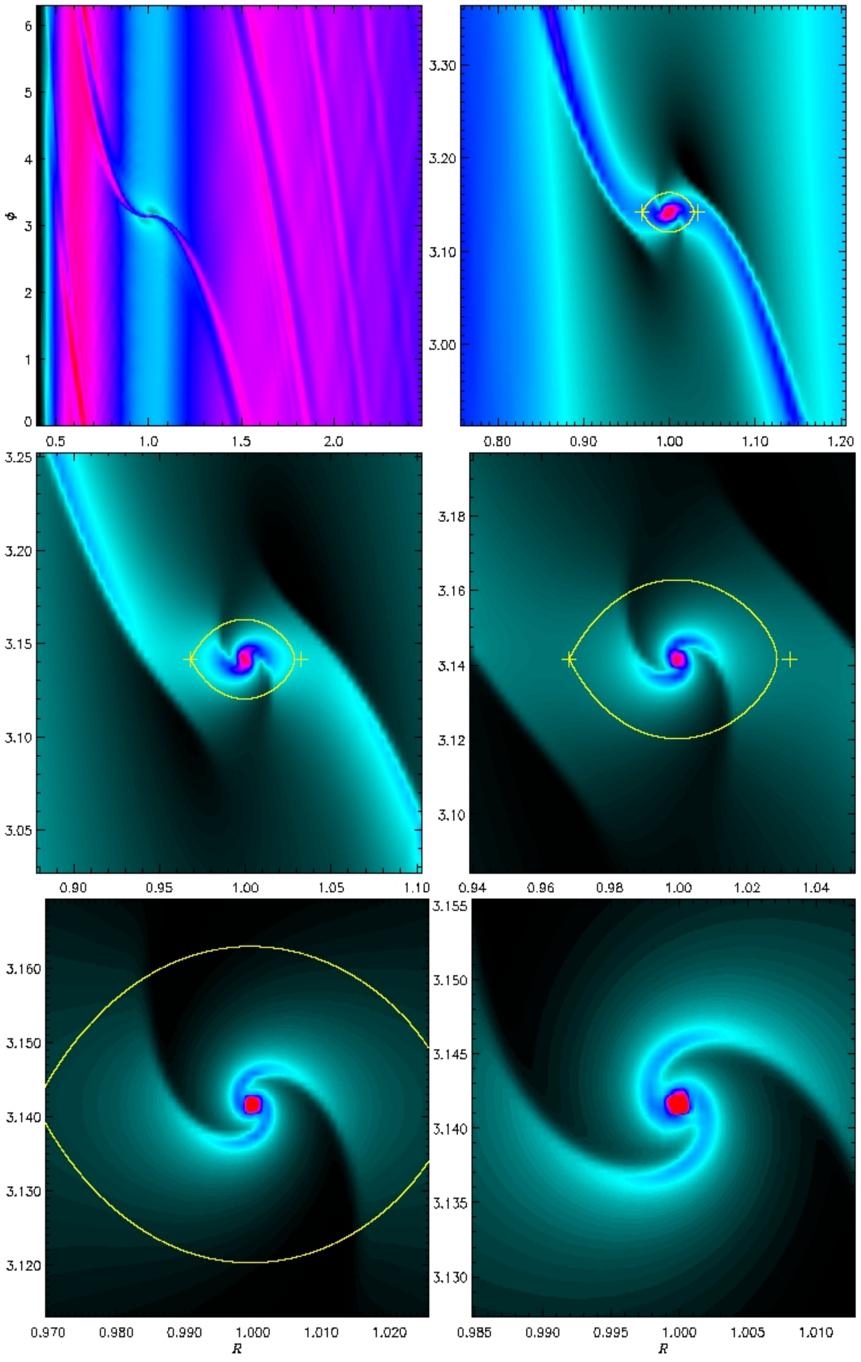
$\varepsilon = 0$ explicit solution (time step limitation); ε different from 0: Implicit solution (system of nonlinear algebraic equations)

Planet-Disk Interaction

Numerical multi-grid simulations
(D´Angelo, Kley, Henning 2003)

Type I (lower mass planets)
Spiral density waves

Type II (higher mass planets)
Creation of a gap



First numerical solution – 1 D (Larson 1969)

1 M_{sun} , $\varrho(t=0) = 10^{-19}$ g cm^{-3}, T = 10 K

Result: Non-homologous evolution (density in outer regions ~ r^{-2})

After free-fall time: Formation of hydrostatic core and free-falling envelope (density ~ $r^{-3/2}$)

At a density of 10^{12} to 10^{13} cm^{-3} $T_c = 100$ K

Contraction of the core until 2000 K is reached and density in center 10^{17} cm^{-3}

Dissociation of molecular hydrogen (endothermic reaction; central region starts to collapse again)

Formation of a second hydrostatic core at central density: 10^{23} cm^{-3} and $T_c = 10^4$ -10^5 K (core still accretes matter which goes through shock front)

At the end: R = 2.1 R_{sun}, L = 1.5 L_{sun} (Confirmed by Winkler & Newman: 2 R_{sun}, 1.0 L_{sun})

After Larson (1969)

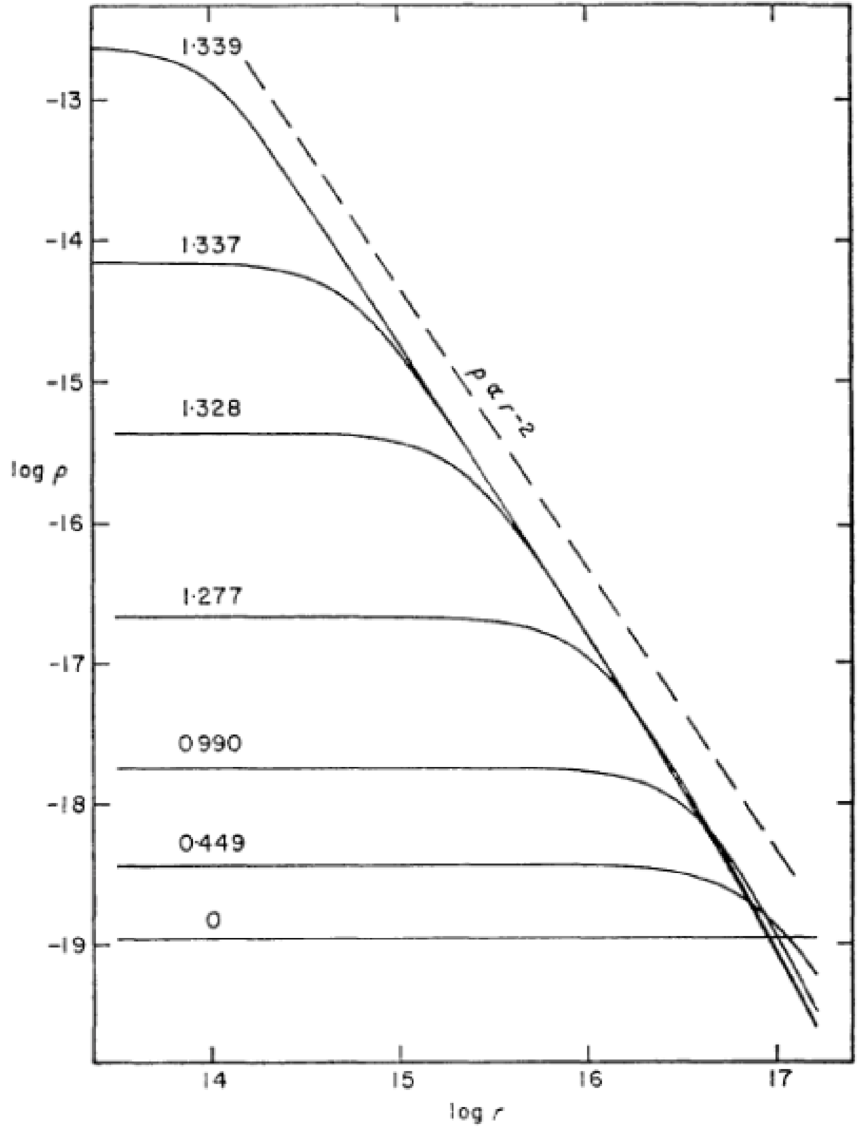
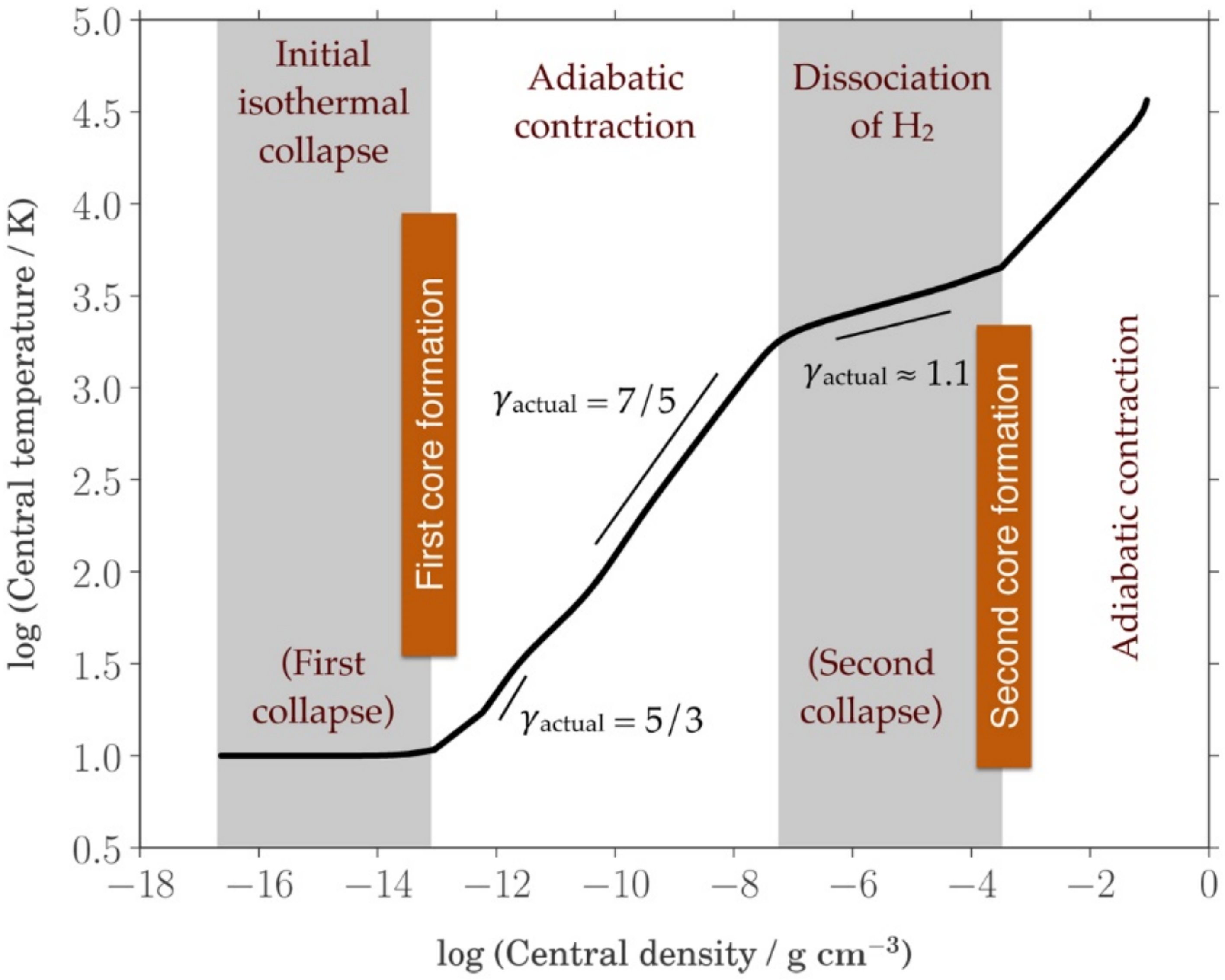


FIG. 1. *The variation with time of the density distribution in the collapsing cloud (CGS units). The curves are labelled with the times in units of* 10^{13} *s since the beginning of the collapse. Note that the density distribution closely approaches the form* $\rho \propto r^{-2}$.

Thermal Evolution of Cloud (1D)

Bhandare
et al. 2018



- Cloud collapses under own gravity (otically thin, isothermal)
- Then cloud becomes optically thick /heats up

Thermal Evolution in 1D

Cloud becomes optically thick at about 10^{-10} g/cm^3; this takes about 10^4 yrs
$R_c = 3$ au, $M_c = 3 \times 10^{-2}\ M_{sun}$

Can we make a temperature estimate? Remember: W = -2 U (time average)

$-GM^2/R = -3\ M\ R_G\ T/\mu$

This means: $T = \mu/3\ R_G\ (GM/R) = 850\ K\ (M/5 \times 10^{-2}\ M_{sun})\ (R/5\ au)^{-1}$

Once T = 2000 K the H_2 molecule begins to dissociate
($\gamma_{eff} = 1.1$ smaller than 4/3 which is the value for stability)

Note: Thermal energy per H_2 molecule is small compared to dissociation energy
($U = 3/2\ R_G\ T/\mu\ M$; Energy/molecule = 0.74 eV; Dissociation energy; 4.48 eV)

Compression energy goes into dissociation! No longer increase of T.

After all H_2 is dissociated: Second hydrostatic core ($R=1.8 \times 10^{-2}$ au; $M=4.6 \times 10^{-3}\ M_{sun}$

Accretion of gas onto protostar

Gas reaches star with free-fall speed which causes an accretion shock front ($T > 10^6$ K; UV and X-rays to be expected)

$$L_{acc} = G\, M_*/R_*\, (dM/dt)$$

$$= 61\, L_{sun}\, (dM/dt/\, 10^{-5}\, M_{sun}/yr)\, (M_*/1\, M_{sun})\, (R_*/5\, R_{sun})^{-1}$$

Additional energy from contraction and early nuclear fusion are negligible compared to L_{acc} for low- to intermediate-mass stars

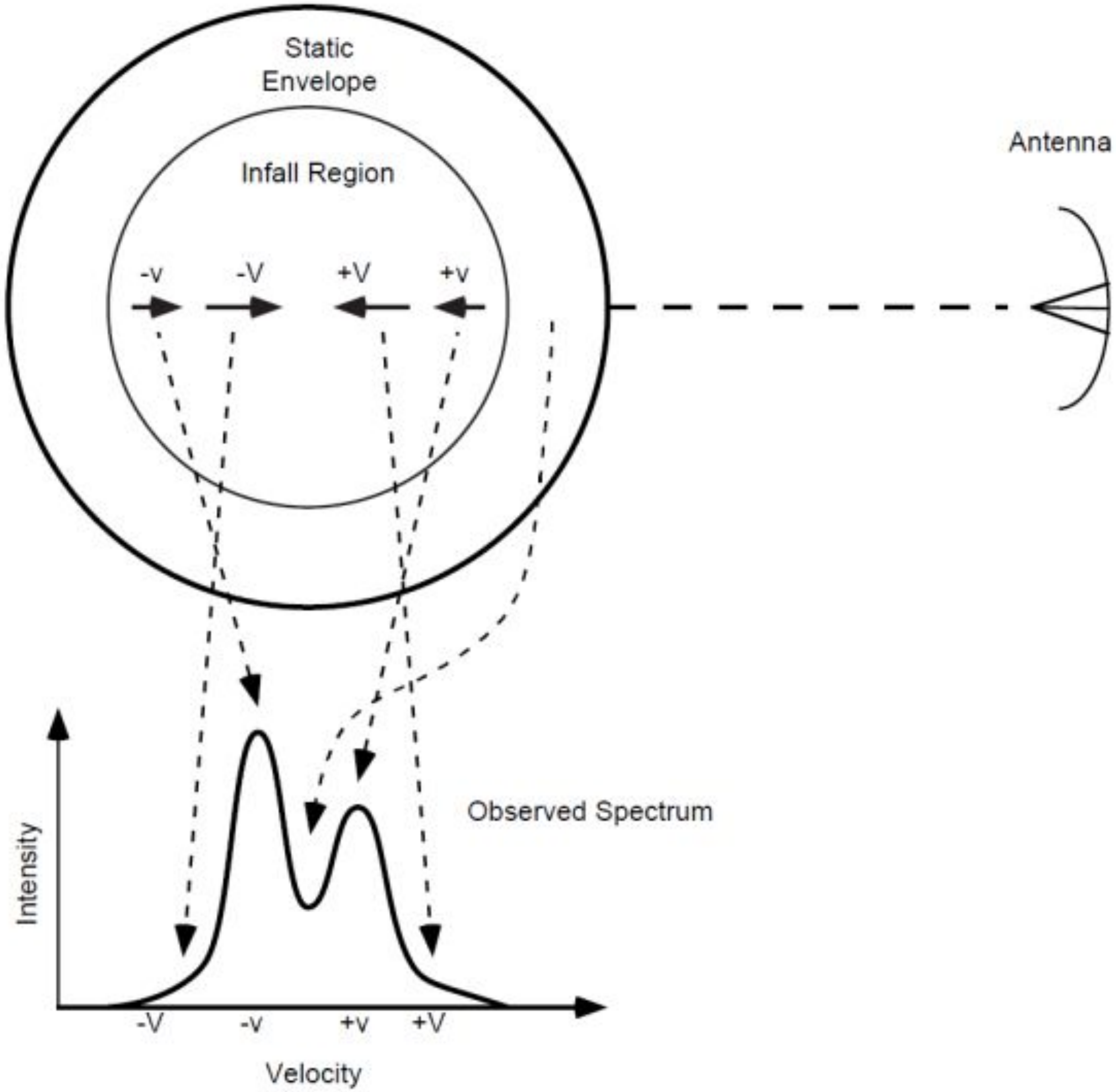
Definition: Low-mass protostar

Mass gaining star with L from accretion shock surrounded by an envelope

- Opacity gap (no dust)
- Inner dust sublimation radius (at about 1 au)
- Effective warm radiating surface observable at mid-IR wavelengths („dust photosphere“) (few au)
- Outer optically thin envelope

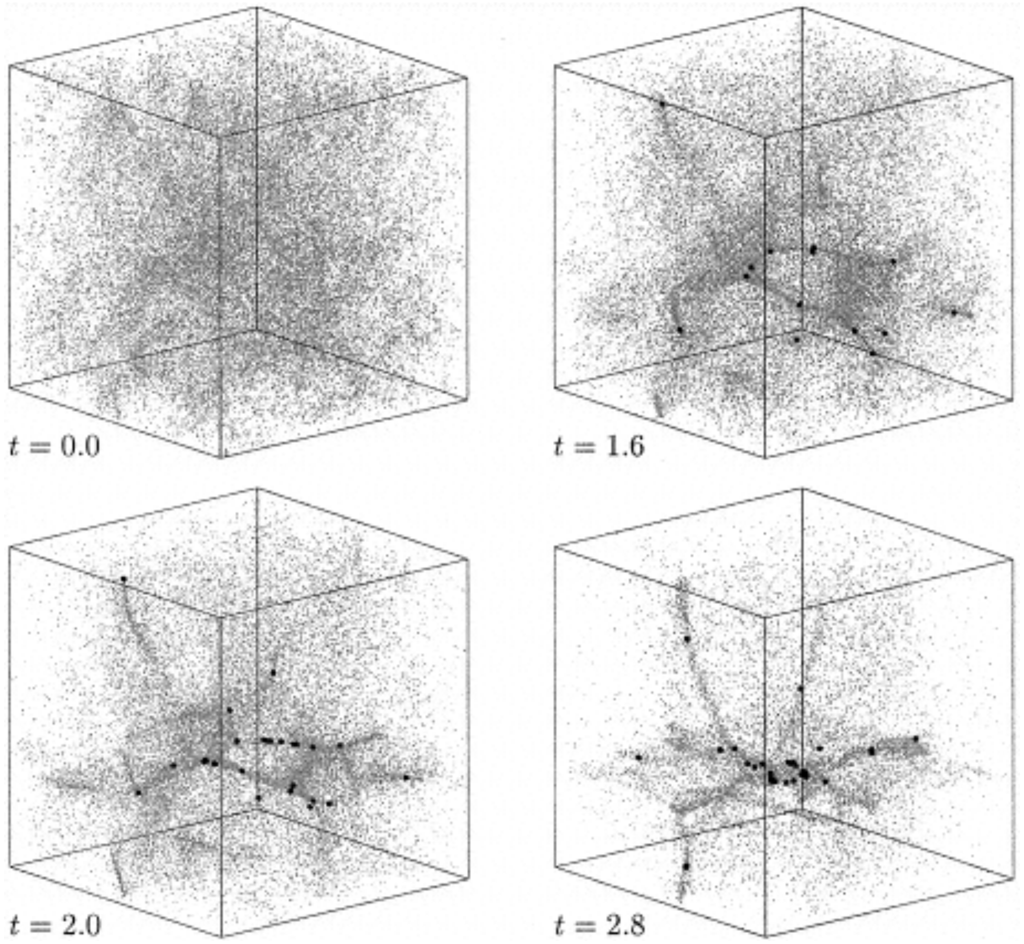
Observing the Collapse of a Cloud

Evans (1999)



Fragmentation of a Cloud (SPH Simulation)

Klessen et al.
(1998)



- 222 Jeans masses
- Spectrum of Gaussian fluctuations