A TABLE OF PISANO PERIOD LENGTHS

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Abstract. A Pisano period is the period of an integer sequence which is obtained by reducing each term of a primary sequence modulo some integer $m \geq 1$. Each primary sequence which obeys a linear homogeneous recurrence has such periods for all $m \geq 1$. The manuscript tabulates the lengths of these periods indexed by m for a small subset of the OEIS sequences.

1. Variants of the Main Theme

The Pisano period is defined as the length of the period of the sequence obtained by reading the Fibonacci sequence [19, A000045] modulo m. The generalized Fibonacci sequences have been investigated in similar ways [8, 3]. The multivariate linear recurrences [2] might be studied for multivariate moduli.

2. Linear Homogeneous Recurrences

2.1. **Definition.** We define a class of sequences $a(n)$ which obey a homogeneous linear recurrence of some depth r, and are essentially characterized by the expansion coefficients c in their linear recurrence:

Definition 1. *(Homogeneous linear recurrence)*

$$a(n) = \sum_{i=1}^{r} c_i^{(0)} a(n-i). \tag{1}$$

It is often advantageous to write this down for the shortest (minimum) order r [12, 17, 11, 24]. The basic properties have been thouroughly studied [4, 22, 23, 9, 20]. There are connections to solving diophantine equations [14], random number generators [10, 13], primality tests [1] aspects of solving the recurrences [6, 7, 16], or transforming other formats to linear recurrences [15, 21].

2.2. **Telescoping.** Telescoping by replacing the coefficient $a(n-1)$ on the right hand side by use of the very same recurrence

$$a(n-1) = \sum_{i=1}^{r} c_i^{(0)} a(n-1-i) \tag{2}$$

transforms (1) into

$$a(n) = \sum_{i=1}^{1} c_i^{(0)} a(n-i) + \sum_{i=2}^{r} c_i^{(0)} a(n-i) = c_1^{(0)} \sum_{i=1}^{r} c_i^{(0)} a(n-1-i) + \sum_{i=2}^{r} c_i^{(0)} a(n-i) \equiv \sum_{i=1}^{r} c_i^{(1)} a(n-1-i) \tag{3}$$

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$$= \sum_{i=1}^{r} c_i^{(0)} c_1^{(0)} a(n-1-i) + \sum_{i=1}^{r-1} c_{i+1}^{(0)} a(n-1-i). \tag{4}$$

(We don't actually care writing down the maximum depth of the telescoping which depends on n being sufficiently large.) This defines constants $c_i^{(1)}$,

$$c_i^{(1)} = c_i^{(0)} c_1^{(0)} + c_{i+1}^{(0)} \tag{5}$$

where it's useful to extend the definition

$$c_i^{(s)} \equiv 0, \quad i > r, \quad \text{any} s \geq 0. \tag{6}$$

Iteration of this procedure deepens the order of the recurrence:

$$a(n) \equiv \sum_{i=1}^{r} c_i^{(s)} a(n-s-i). \tag{7}$$

The constants follow from a multiply-shift operation

$$c_i^{(s)} = c_1^{(s-1)} c_i^{(0)} + c_{i+1}^{(s-1)}, \quad s \geq 1, \quad 1 \leq i \leq r, \tag{8}$$

which can be written in matrix format as

$$\begin{pmatrix} c_1^{(s)} \\ c_2^{(s)} \\ \vdots \\ c_r^{(s)} \end{pmatrix} = \begin{pmatrix} c_1^{(0)} & 1 & 0 & 0 & 0 \\ c_2^{(0)} & 0 & 1 & 0 & 0 \\ c_3^{(0)} & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{r-1}^{(0)} & 0 & 0 & 0 & 1 \\ c_r^{(0)} & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_1^{(s-1)} \\ c_2^{(s-1)} \\ \vdots \\ c_r^{(s-1)} \end{pmatrix}. \tag{9}$$

Remark 1. *Diagonalization of this square matrix would allow to write* $c_i^{(s)}$ *as some matrix power multiplied by* $c_i^{(0)}$. *The determinant*

$$\begin{vmatrix} c_1^{(0)} - \lambda & 1 & 0 & 0 & 0 \\ c_2^{(0)} & -\lambda & 1 & 0 & 0 \\ c_3^{(0)} & 0 & -\lambda & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{r-1}^{(0)} & 0 & 0 & -\lambda & 1 \\ c_r^{(0)} & 0 & 0 & 0 & -\lambda \end{vmatrix} \tag{10}$$

$$= (-1)^{r-1}[(c_1^{(0)} - \lambda)\lambda^{r-1} + c_2^{(0)}\lambda^{r-2} + c_3^{(0)}\lambda^{r-3} + \cdots + c_r^{(0)}]$$

allows computation of the eigenvalues λ. *Up to signs, this equals the companion polynomial* $f(\lambda)$, *defined by*

$$f(x) = x^m - c_{m-1}x^{m-1} - \ldots c_0. \tag{11}$$

The recurrence (8) has a markovian property that the r-tuple at step s depends on the values of the previous step and on the values $c_i^{(0)}$ that define the recurrence. Because the new values are obtained by multiplication which can be reduced modulo k, a certain vector modulo k is followed uniquely by another vector modulo k; since there is only a finite set of r^k different vectors, this reduced vector must reach a state encountered at some previous s within at most r^k steps; both conditions combined ensure that the vector $c_i^{(s)}(modk)$ with r elements is periodic in the upper index s.

Graphically speaking, the vector of r previous values on the right hand side of (1), each reduced modulo some k, defines a point in a cube of edge length k, and computing $a(n)$, also reduced modulo k, is one step of a walk inside that cube. The Pisano periods are closed walks (cycles), and the initial conditions of the a at low indices fix to which of these non-intersecting cycles that walk belongs. Obviously, the number of non-intersecting cycles in that cube of fixed dimesion is limited, so the family of sequences with the same recurrence cofficients c_i and different initial values leads to a finite number of Pisano periods.

3. Pisano Periods

The question is: Does a finite period length p_k exist such that

$$a(n+p_k) = a(n) (\bmod k) \quad \forall n \geq ...? \tag{12}$$

The answer is yes and known [18]. Because for any s (in the telescoped recurrence)

$$a(n) = \sum_{i=1}^{r} c_i^{(0)} a(n-i) = \sum_{i=1}^{r} c_i^{(s)} a(n-i-s) \tag{13}$$

we have for all s and k

$$\sum_{i=1}^{r} c_i^{(0)} a(n-i) = \sum_{i=1}^{r} c_i^{(s)} a(n-i-s) (\bmod k). \tag{14}$$

$$\sum_{i=1}^{r} [c_i^{(0)} a(n-i) - c_i^{(s)} a(n-i-s)] = 0 \, (\bmod k). \tag{15}$$

$$\sum_{i=1}^{r} [c_i^{(0)}(modk) a(n-i)(modk) - c_i^{(s)}(modk) a(n-i-s)(modk)] = 0 \, (\bmod k). \tag{16}$$

Insertion of (12) leads to the *necessary* condition,

$$\sum_{i=1}^{r} [c_i^{(0)}(modk) - c_i^{(p)}(modk)] a(n-i)(modk) = 0 \, (\bmod k). \tag{17}$$

which can be tested (in the sense that the following is a certificate for the necessary condition) by applying the condition to each individual term:

$$c_i^{(0)} - c_i^{(p)} = 0 (\bmod k) \forall 1 \leq i \leq r \tag{18}$$

(Certificate means that satisfying this satisfies the necessary condition, but since the necessary condition is summatory, a dot product of the vector of length r of the c with the subsequence of the actual period build by the a mod k, there are additional ways of satisfying the ncessary condition if the period is providing matching weights. The order of the recurrence r is finite and k is finite, which means that the r-tupel $c_i^{(0)} - c_i^{(p)}$ has at most r^k different values (mod k). Steadily increasing s and testing

$$c_i^{(0)} = c_i^{(s)} (\bmod k) \forall 1 \leq i \leq r \tag{19}$$

there can in principle be the following scenarios:

- the tuples c_i are aperiodic

- the tuples c_i are periodic and the matching condition is fulfilled for s, $s+p_s$, $s+2p_s$,..

The period mentioned here can either include or not include all the r^k different states.

Since (19) is sufficient to show periodicity of the a_n, the periodicity of the $c_i(s)$ (mod k) mentioned above ensures that this s defines a Pisano period length (not necessarily the smallest one). The periodicity of the modular $c_i^{(s)}$ enforces the periodicity of the modular $a(n)$: We have by definition of the c_i:

$$a(n) = \sum_{i=1}^{r} c_i^{(s)} a(n-i-s); \tag{20}$$

$$a(n-s) = \sum_{i=1}^{r} c_i^{(0)} a(n-i-s); \tag{21}$$

the difference

$$a(n-s) - a(n) = \sum_{i=1}^{r} [c_i^{(0)} - c_i^{(s)}] a(n-i-s); \tag{22}$$

and modulo variant under the proposition

$$a(n-s) - a(n) = \sum_{i=1}^{r} 0 \cdot a(n-i-s) (\mathrm{mod} k); \tag{23}$$

q.e.d.

The general case is that the initial sets of $c_i^{(s)}$ (mod k) are preperiodic (transient) for some small s before entering the period, which replaces (19) by the state condition

$$c_i^{(j)} = c_i^{(j+s)} (\mathrm{mod}\, k)\, \forall 1 \le i \le r, \tag{24}$$

so

$$a(n) = \sum_i c_i^j a(n-j-i) = \sum_i c_i^{j+s} a(n-j-s-i). \tag{25}$$

Remark 2. *If $a(n)$ is a polynomial of degree l with coefficients β, the Pisano period length is limited, as seen by the binomial expansion,*

$$a(n) = \sum_{j=0}^{l} \beta_j n^j \Rightarrow a(n+k) = \sum_{j=0}^{l} \beta_j (n+k)^j = \sum_{j=0}^{l} \beta_j \sum_{t=0}^{j} \binom{j}{t} n^t k^{j-t} \tag{26}$$

$$\therefore a(n+k) \equiv \sum_{j=0}^{l} \beta_j \sum_{t=j}^{j} \binom{j}{t} n^t k^{j-t} = \sum_{j=0}^{l} \beta_j n^j = a(n) \pmod{k}, \tag{27}$$

which leads to $p_k \le k$.

4. Main Table

The following table shows Pisano periods for some sequences characterized by their number in the Encyclopedia of Integer Sequenes [19] in the first column, the associated Pisano period length in the same notation in the second column, the coeficients $c_i^{(0)}$ in square brackets in the third column ("signature"), and the $p_1, p_2, p_3, \ldots$ for the initial indices in the next square bracket.

A000302	[4]	[1, 1, 1, 1, 2, 1, 3, 1, 3, 2, 5, 1, 6, 3, 2, 1, 4, 3, 9, 2,]
A141413	[-3]	[1, 1, 1, 1, 4, 1, 3, 2, 1, 4, 10, 1, 6, 3, 4, 4, 16, 1, 9, 4,]
A003665	[2, 8]	[1, 1, 1, 1, 4, 1, 6, 1, 1, 4, 5, 1, 12, 6, 4, 1, 8, 1, 9, 4,]
A122803	[-2]	[1, 1, 1, 1, 4, 1, 6, 1, 3, 4, 5, 1, 12, 6, 4, 1, 8, 3, 9, 4,]
A015564	[7, 6]	[1, 1, 1, 1, 12, 1, 4, 2, 3, 12, 15, 1,168, 4, 12, 4,288, 3, 18, 12,]
A083858	[3, 6]	[1, 1, 1, 1, 12, 1, 8, 1, 1, 12,110, 1,168, 8, 12, 2, 16, 1,360, 12,]
A026150	[2, 2]	[1, 1, 1, 1, 24, 1, 48, 1, 3, 24, 10, 1, 12, 48, 24, 1,144, 3,180, 24,]
A087451	[1, 6]	[1, 1, 1, 2, 4, 1, 6, 2, 3, 4, 5, 2, 12, 6, 4, 4, 16, 3, 18, 4,]
A015441	[1, 6]	[1, 1, 1, 2, 20, 1, 6, 2, 3, 20, 5, 2, 12, 6, 20, 4, 16, 3, 18, 20,]
A002533	[2, 5]	[1, 1, 1, 4, 4, 1, 24, 4, 3, 4,120, 4, 56, 24, 4, 8,288, 3, 18, 4,]
A000351	[5]	[1, 1, 2, 1, 1, 2, 6, 2, 6, 1, 5, 2, 4, 6, 2, 4, 16, 6, 9, 1,]
A015580	[9, 4]	[1, 1, 2, 1, 3, 2, 48, 2, 6, 3, 10, 2, 42, 48, 6, 4, 24, 6,360, 3,]
A000225	[3, -2]	[1, 1, 2, 1, 4, 2, 3, 1, 6, 4, 10, 2, 12, 3, 4, 1, 8, 6, 18, 4,]
A000051	[3, -2]	[1, 1, 2, 1, 4, 2, 3, 1, 6, 4, 10, 2, 12, 3, 4, 1, 8, 6, 18, 4,]
A046717	[2, 3]	[1, 1, 2, 1, 4, 2, 6, 4, 2, 4, 10, 2, 6, 6, 4, 8, 16, 2, 18, 4,]
A077966	[0, -2]	[1, 1, 2, 1, 8, 2, 12, 1, 6, 8, 10, 2, 24, 12, 8, 1, 16, 6, 18, 8,]
A077957	[0, -2]	[1, 1, 2, 1, 8, 2, 12, 1, 6, 8, 10, 2, 24, 12, 8, 1, 16, 6, 18, 8,]
A083099	[2, 6]	[1, 1, 2, 1, 12, 2, 7, 1, 6, 12, 60, 2,168, 7, 12, 1,288, 6, 18, 12,]
A015540	[5, 6]	[1, 1, 2, 2, 2, 2, 14, 2, 2, 2, 10, 2, 12, 14, 2, 2, 16, 2, 18, 2,]
A014551	[1, 2]	[1, 1, 2, 2, 4, 2, 6, 2, 6, 4, 10, 2, 12, 6, 4, 2, 8, 6, 18, 4,]
A015521	[3, 4]	[1, 1, 2, 2, 10, 2, 6, 2, 6, 10, 10, 2, 6, 6, 10, 2, 4, 6, 18, 10,]
A001834	[4, -1]	[1, 1, 2, 4, 3, 2, 8, 4, 6, 3, 10, 4, 12, 8, 6, 8, 18, 6, 5, 12,]
A077959	[0, 0, -2]	[1, 1, 3, 1, 12, 3, 18, 1, 9, 12, 15, 3, 36, 18, 12, 1, 24, 9, 27, 12,]
A085750	[-4, -4]	[1, 1, 3, 1, 20, 3, 42, 1, 9, 20, 55, 3,156, 42, 60, 1,136, 9,171, 20,]
A021006	[2, 2]	[1, 1, 3, 1, 24, 3, 48, 1, 9, 24, 10, 3, 12, 48, 24, 1,144, 9,180, 24,]
A002605 A175289	[2, 2]	[1, 1, 3, 1, 24, 3, 48, 1, 9, 24, 10, 3, 12, 48, 24, 1,144, 9,180, 24,]
A028859	[2, 2]	[1, 1, 3, 1, 24, 3, 48, 1, 9, 24, 10, 3, 12, 48, 24, 1,144, 9,180, 24,]
A015535	[5, 2]	[1, 1, 3, 2, 8, 3, 48, 2, 3, 8,110, 6,168, 48, 24, 4, 8, 3, 45, 8,]
A016116	[0, 2]	[1, 1, 4, 1, 8, 4, 6, 1, 12, 8, 20, 4, 24, 6, 8, 1, 16, 12, 36, 8,]
A077957	[0, 2]	[1, 1, 4, 1, 8, 4, 6, 1, 12, 8, 20, 4, 24, 6, 8, 1, 16, 12, 36, 8,]

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A015584                 [9, 8]  [  1,  1,  4,  1, 24,  4,  6,  1,  4, 24, 10,  4, 12,  6, 24,  1,144,  4, 15, 24,]
A104934                 [3, 2]  [  1,  1,  4,  1, 24,  4, 48,  1, 12, 24, 30,  4, 12, 48, 24,  1,272, 12, 18, 24,]
A055099                 [3, 2]  [  1,  1,  4,  1, 24,  4, 48,  1, 12, 24, 30,  4, 12, 48, 24,  2,272, 12, 18, 24,]
A007482                 [3, 2]  [  1,  1,  4,  1, 24,  4, 48,  2, 12, 24, 30,  4, 12, 48, 24,  4,272, 12, 18, 24,]
A132429           [-1, -1, -1]  [  1,  1,  4,  2,  4,  4,  4,  4,  4,  4,  4,  4,  4,  4,  4,  4,  4,  4,  4,  4,]
A054490                [6, -1]  [  1,  1,  4,  2,  6,  4,  3,  2, 12,  6, 12,  4, 14,  3, 12,  2,  8, 12, 20,  6,]
A094359             [-1, 4, 4]  [  1,  1,  6,  1,  4,  6,  6,  2, 18,  4, 10,  6, 12,  6, 12,  2,  8, 18, 18,  4,]
A078008                 [1, 2]  [  1,  1,  6,  1,  4,  6,  6,  2, 18,  4, 10,  6, 12,  6, 12,  2,  8, 18, 18,  4,]
A015443                 [1, 8]  [  1,  1,  6,  1, 24,  6, 16,  1,  6, 24,110,  6, 56, 16, 24,  2, 16,  6, 60, 24,]
A077917            [-1, 4, -2]  [  1,  1,  6,  1, 24,  6, 48,  1, 18, 24,  5,  6, 12, 48, 24,  1,144, 18,180, 24,]
A048573                 [1, 2]  [  1,  1,  6,  2,  2,  6,  6,  2, 18,  2, 10,  6, 12,  6,  6,  2,  8, 18, 18,  2,]
A001045 A175286         [1, 2]  [  1,  1,  6,  2,  4,  6,  6,  2, 18,  4, 10,  6, 12,  6, 12,  2,  8, 18, 18,  4,]
A057087                 [4, 4]  [  1,  1,  8,  1,  3,  8,  6,  1, 24,  3,120,  8, 21,  6, 24,  1, 16, 24,360,  3,]
A015568                [7, 10]  [  1,  1,  8,  1,  4,  8, 12,  1, 24,  4,  5,  8, 21, 12,  8,  1, 16, 24, 45,  4,]
A009545                [2, -2]  [  1,  1,  8,  1,  4,  8, 24,  1, 24,  4, 40,  8, 12, 24,  8,  1, 16, 24, 72,  4,]
A090132               [-2, -2]  [  1,  1,  8,  1,  4,  8, 24,  1, 24,  4, 40,  8, 12, 24,  8,  1, 16, 24, 72,  4,]
A108520               [-2, -2]  [  1,  1,  8,  1,  4,  8, 24,  1, 24,  4, 40,  8, 12, 24,  8,  1, 16, 24, 72,  4,]
A009116               [-2, -2]  [  1,  1,  8,  1,  4,  8, 24,  1, 24,  4, 40,  8, 12, 24,  8,  1, 16, 24, 72,  4,]
A099087                [2, -2]  [  1,  1,  8,  1,  4,  8, 24,  1, 24,  4, 40,  8, 12, 24,  8,  1, 16, 24, 72,  4,]
A078069               [-2, -2]  [  1,  1,  8,  1,  4,  8, 24,  1, 24,  4, 40,  8, 12, 24,  8,  1, 16, 24, 72,  4,]
A015537                 [5, 4]  [  1,  1,  8,  1,  4,  8, 48,  1, 24,  4, 40,  8, 42, 48,  8,  2, 72, 24,360,  4,]
A006495                [2, -5]  [  1,  1,  8,  1,  4,  8, 48,  4, 24,  4, 60,  8, 12, 48,  8,  8, 16, 24, 90,  4,]
A063727                 [2, 4]  [  1,  1,  8,  1,  5,  8, 48,  1, 24,  5, 10,  8, 42, 48, 40,  1, 72, 24, 18,  5,]
A072265                 [1, 4]  [  1,  1,  8,  1,  6,  8, 48,  2, 24,  6,120,  8, 12, 48, 24,  4,  8, 24, 18,  6,]
A006131                 [1, 4]  [  1,  1,  8,  1,  6,  8, 48,  2, 24,  6,120,  8, 12, 48, 24,  4,136, 24, 18,  6,]
A164539                 [2, 7]  [  1,  1,  8,  1, 24,  8,  3,  2, 24, 24, 15,  8,168,  3, 24,  2,  4, 24,120, 24,]
A006012                [4, -2]  [  1,  1,  8,  1, 24,  8,  6,  1, 24, 24,120,  8,168,  6, 24,  1,  8, 24,360, 24,]
A007070                [4, -2]  [  1,  1,  8,  1, 24,  8,  6,  1, 24, 24,120,  8,168,  6, 24,  1,  8, 24,360, 24,]
A090591                [2, -8]  [  1,  1,  8,  1, 24,  8,  7,  1, 24, 24, 10,  8, 56,  7, 24,  1,144, 24,120, 24,]
A078020                [1, -2]  [  1,  1,  8,  1, 24,  8, 21,  2, 24, 24, 10,  8,168, 21, 24,  2,144, 24,360, 24,]
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A110512                [-1, -2]  [  1,  1,  8,  1, 24,  8, 42,  1, 24, 24, 10,  8,168, 42, 24,  2,144, 24,360, 24,]
A001607                [-1, -2]  [  1,  1,  8,  1, 24,  8, 42,  2, 24, 24, 10,  8,168, 42, 24,  4,144, 24,360, 24,]
A099843                 [-4, 1]  [  1,  1,  8,  2, 20,  8, 16,  4,  8, 20, 10,  8, 28, 16, 40,  8, 12,  8,  6, 20,]
A107920                 [1, -2]  [  1,  1,  8,  2, 24,  8, 21,  2, 24, 24, 10,  8,168, 21, 24,  4,144, 24,360, 24,]
A006355                  [1, 1]  [  1,  1,  8,  3, 20,  8, 16,  6, 24, 20, 10, 24, 28, 16, 40, 12, 36, 24, 18, 60,]
A123335                 [-2, 1]  [  1,  1,  8,  4, 12,  8,  6,  4, 24, 12, 24,  8, 28,  6, 24,  8, 16, 24, 40, 12,]
A048694                  [2, 1]  [  1,  1,  8,  4, 12,  8,  6,  4, 24, 12, 24,  8, 28,  6, 24,  8, 16, 24, 40, 12,]
A001333                  [2, 1]  [  1,  1,  8,  4, 12,  8,  6,  4, 24, 12, 24,  8, 28,  6, 24,  8, 16, 24, 40, 12,]
A077974             [-1, 0, -2]  [  1,  1, 13,  1,124, 13, 48,  2, 39,124,665, 13,2196, 48,1612,  4, 72, 39, 18,124,]
A001644 A106293       [1, 1, 1]  [  1,  1, 13,  4, 31, 13, 48,  8, 39, 31, 10, 52,168, 48,403, 16, 96, 39,360,124,]
A000213               [1, 1, 1]  [  1,  1, 13,  4, 31, 13, 48,  8, 39, 31,110, 52,168, 48,403, 16, 96, 39,360,124,]
A003462                 [4, -3]  [  1,  2,  1,  2,  4,  2,  6,  4,  1,  4,  5,  2,  3,  6,  4,  8, 16,  2, 18,  4,]
A014983                 [-2, 3]  [  1,  2,  1,  4,  4,  2,  3,  8,  1,  4, 10,  4,  6,  6,  4, 16, 16,  2,  9,  4,]
A015591                 [10, 9]  [  1,  2,  1,  4,  4,  2, 48,  8,  1,  4, 10,  4, 84, 48,  4, 16,272,  2,360,  4,]
A038754                  [0, 3]  [  1,  2,  1,  4,  8,  2, 12,  4,  1,  8, 10,  4,  6, 12,  8,  8, 32,  2, 36,  8,]
A015530                  [4, 3]  [  1,  2,  1,  4, 24,  2, 21,  4,  3, 24, 40,  4, 84, 42, 24,  8,288,  6, 18, 24,]
A000034 A130130          [0, 1]  [  1,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,  2,]
A001075                 [4, -1]  [  1,  2,  2,  4,  3,  2,  8,  4,  6,  6, 10,  4, 12,  8,  6,  8, 18,  6,  5, 12,]
A015518                  [2, 3]  [  1,  2,  2,  4,  4,  2,  6,  8,  2,  4, 10,  4,  6,  6,  4, 16, 16,  2, 18,  4,]
A015574                  [8, 3]  [  1,  2,  2,  4,  4,  2, 24,  4,  6,  4,120,  4, 84, 24,  4,  8, 16,  6,171,  4,]
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```

A first result is that the generalized Fibonacci sequences of the signature $[1, 1]$,

$$a(n) = a(n-1) + a(n-2), \quad a(0) = \alpha_0, \quad a(1) = \alpha_1, \tag{28}$$

share many period lengths. This is a result of an inheritance of periods of the following type [5]: the Fibonacci numbers [19, A000045] are

$$F(n) = F(n-1) + F(n-2), \quad F(0) = 1, \quad F(1) = 1 \tag{29}$$

with generating function

$$g(x) = \frac{x}{1 - x - x^2}. \tag{30}$$

The generalized Fibonacci sequences are linear combinations

$$a(n) = (\alpha_1 - \alpha_0)F(n) + \alpha_0 F(n+1) \tag{31}$$

with generating function

$$g(x) = \frac{\alpha_0 + (\alpha_1 - \alpha_0)x}{1 - x - x^2} \tag{32}$$

and if on the right hand side of this equation $F(n) = F(n + p_k) \pmod{k}$ and $F(n+1) = F(n+1+p_k) \pmod{k}$, then the left hand side is also periodic with period length $p_k \pmod{k}$—and for some k the period may be shorter. In this spirit, the sequences with the simplest 1-term numerator in their generating function are base sequences, the sequences with multi-term numerators are linear superpositions of these (with cofficients to be read off the numerators), and the Pisano period lengths of the derived sequences are never larger than the period of the basic sequence (and therefore divisors of that period [4]).

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