EQUIVALENCE OF A020483 AND A054906

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Abstract. A claim by Labos Elemer is that finding primes p which have a distance $2n$ to another prime q is essentially the same as finding x that solve $\sigma(x+2n) = \sigma(x) + 2n$. We attempt to proof this conjecture.

1. Definitions

Definition 1. *Sequence* [1, A020483] *is defined by the smallest prime* p *which has a distance* $2n$ *to another prime* q*:*

$$p + 2n = q. \tag{1}$$

Remark 1. *This does not mean that the prime gap after* p *is* $2n$ *because there may be other primes between* p *and* q*, see* [1, A000230].

Definition 2. *Sequence* [1, A054906] *is defined by the smallest integer solution* x *to*

$$\sigma(x + 2n) = \sigma(x) + 2n \tag{2}$$

where

$$\sigma(n) \equiv \sum_{d|n} d \tag{3}$$

is the sums-of-divisors function [1, A000203].

This is multiplicative with

$$\sigma(p^e) = \frac{p^{e+1} - 1}{p - 1} = 1 + p + p^2 + \cdots + p^e. \tag{4}$$

Definition 3. *(Sums of proper divisors)*

$$\bar{\sigma}(n) \equiv \sum_{d|n, 1<d<n} d \tag{5}$$

is the sum of the proper divisors of n [1, A048050].

For all relevant cases (i.e., $x + 2n > 1$), (2) is equivalent to

$$\bar{\sigma}(x + 2n) = \bar{\sigma}(x). \tag{6}$$

Heuristically we find that the sequence of the p is the same as the sequence of the x for at least $n \le 1,560,000$, and this here shows first steps to proof the conjecture:

Conjecture 1. *Sequences A020483 and A054906 are the same.*

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2. Forward Part of the Proof

Supposed a solution p to (1) has been found, this p solves (2) with $x = p$, because then clearly

$$\sigma(x+2n) = \sigma(p+2n) = \sigma(q) = 1+q = 1+p+2n = \sigma(p)+2n = \sigma(x)+2n \tag{7}$$

due $\sigma(p) = 1+p$ for primes p. Given such p we thus have found an upper bound to x, therefore

Theorem 1.

$$A054906(n) \le A020483(n). \tag{8}$$

It remains to show vice versa that solutions x are upper bounds to solutions p. This is incompletely shown in the next section.

3. Attempted Reverse Part of the Proof

3.1. **Prime x.** The solutions x to (2) may either be prime or composite—see [1, A054905] for the smallest composite x.

Theorem 2. *If a solution x to (2) is prime, then $x+2n$ is also prime.*

Proof. If the solution x to (2) is prime, then

$$\sigma(x) + 2n = 1 + x + 2n. \tag{9}$$

In conjunction, $x+2n$ may either be prime or composite (discarding the marginal case where $x+2n = 1$ is neither prime nor composite). If $x+2n$ is composite, $\sigma(x+2n) > 1+x+2n$ because then $x+2n$ must have at least one divisor besides 1 and $x+2n$ that contributes to $\sigma(x+2n)$.[1] This inequality means to (9) that $\sigma(x)+2n = 1+x+2n < \sigma(x+2n)$, so x cannot solve (2), therefore the assumption that $x+2n$ is composite cannot be true. □

Theorem 2 says that prime solutions x to (2) are solutions to (1). So the smallest x in the sorted listed of prime solutions to (2) is an upper bound to the n-th entry of sequence A020483. This statement is a kind of incomplete reversal of the sign in (1); it is incomplete because the list of all solutions to (2) might also start with a *composite* x. To complete the proof we need to show that the smallest x is always prime (never composite), so conjecture 1 is proven if we proof

Conjecture 2. *The smallest solution x to (2) is always prime.*

3.2. **Composite x.** The prime solutions x to (2) are minimalistic in the sense that they solve (6) with both sides becoming zero, whereas for composite x both sides are larger than zero.

References

1. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, http://oeis.org/. MR 1992789 (2004f:11151)

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[1] meaning that [1, A062825] is positive for all composite arguments.